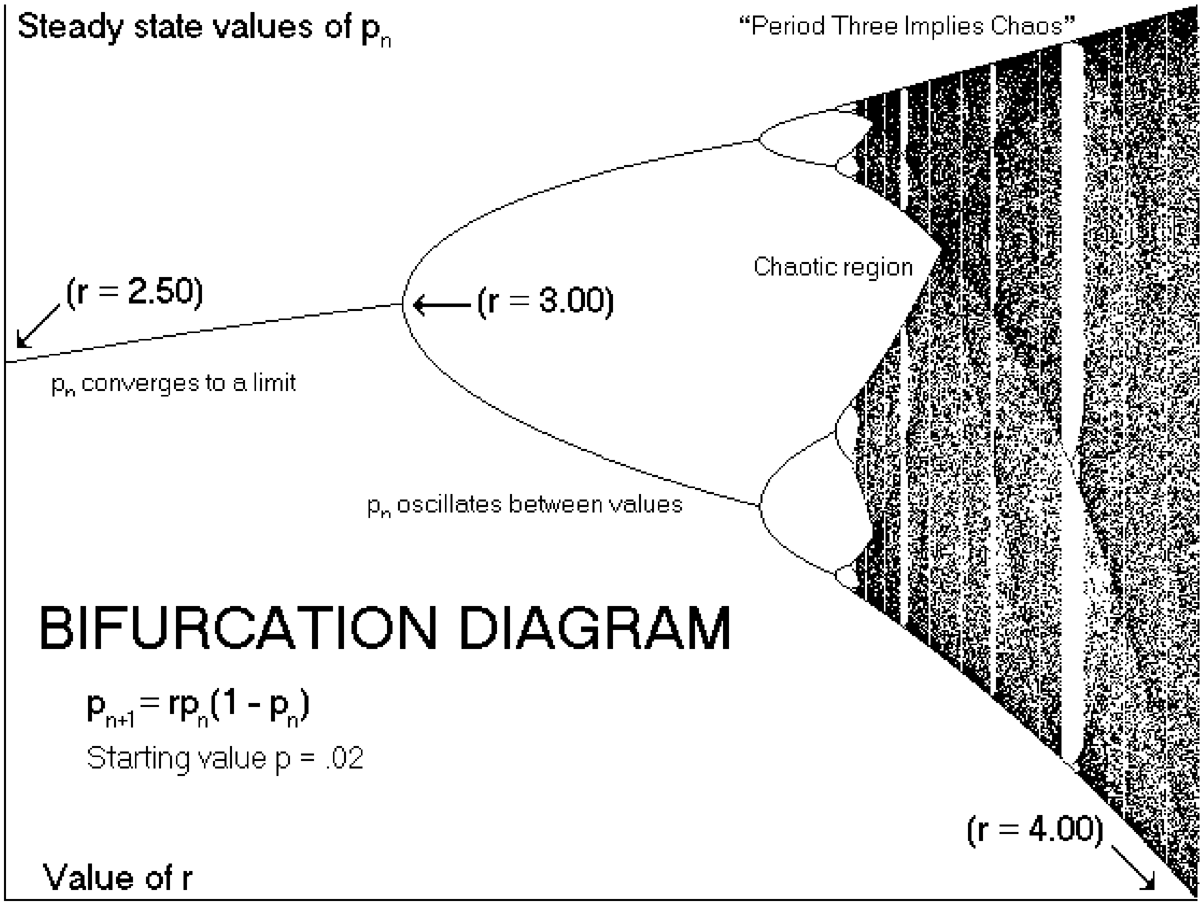

Steady state values of p_n
"Period Three Implies Chaos"
Chaotic region
(r = 2.50)
(r = 3.00)
p_n converges to a limit
p_n oscillates between values
BIFURCATION DIAGRAM
$p_{n+1} = rp_n(1 - p_n)$
Starting value p = .02
(r = 4.00)
Value of r

Chaos Theory and Fractals

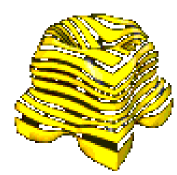
By Jonathan Mendelson and Elana Blumenthal

Outline

- Introduction to Chaos
- Determinism
- Early Chaos
- Edward Lorenz
- Chaos Theory
- Initial Conditions
- Chaos on the Large Scale
- Instability
- Chaos in the Real World
- Complexity
- Fractals
- Benoit Mandelbrot
- Sierpinski's Triangle
- Koch Snowflake
- Mandelbrot Set
- More Complex Fractals
- Chaos Theory and Fractal Links
- Works Cited

Introduction to Chaos

The dictionary definition of chaos is turmoil, turbulence, primordial abyss, and undesired randomness, but scientists will tell you that chaos is something extremely sensitive to initial conditions. Chaos also refers to the question of whether or not it is possible to make good long-term predictions about how a system will act. A chaotic system can actually develop in a way that appears very smooth and ordered.



Determinism

Sir Isaac Newton

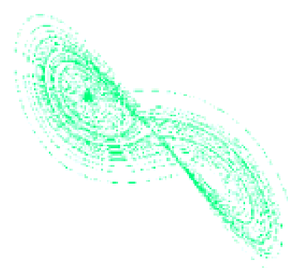
Determinism is the belief that every action is the result of preceding actions. It began as a philosophical belief in Ancient Greece thousands of years ago and was introduced into science around 1500 A.D. with the idea that cause and effect rules govern science. Sir Isaac Newton was closely associated with the establishment of determinism in modern science. His laws were able to predict systems very accurately. They were deterministic at their core because they implied that everything that would occur would be based entirely on what happened right before. The Newtonian model of the universe is often depicted as a billiard game in which the outcome unfolds mathematically from the initial conditions in a pre-determined fashion, like a movie that can be run forwards or backwards in time. Determinism remains as one of the more important concepts of physical science today.

Early Chaos

Ilya Prigogine showed that complex structures could come from simpler ones. This is like order coming from chaos. Henry Adams previously described this with his quote "Chaos often breeds life, when order breeds habit". Henri Poincaré was really the "Father of Chaos [Theory]," however. The planet Neptune was discovered in 1846 and had been predicted from the observation of deviations in Uranus' orbit. King Oscar II of Norway was willing to give a prize to anyone who could prove or disprove that the solar system was stable. Poincaré offered his solution, but when a friend found an error in his calculations, the prize was taken away until he could come up with a new solution that worked. He found that there was no solution. Not even Sir Isaac Newton's laws provided a solution to this huge problem. Poincaré had been trying to find order in a system where there was none to be found.

Edward Lorenz

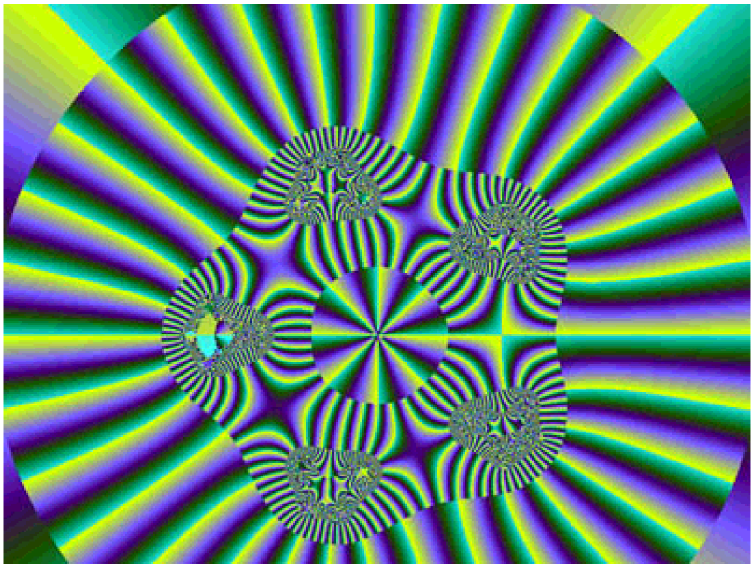
During the 1960's Edward Lorenz was a meteorologist at MIT working on a project to simulate weather patterns on a computer. He accidentally stumbled upon the butterfly effect after deviations in calculations off by thousandths greatly changed the simulations. The Butterfly Effect reflects how changes on the small scale affect things on the large scale. It is the classic example of chaos, as small changes lead to large changes. An example of this is how a butterfly flapping its wings in Hong Kong could change tornado patterns in Texas. Lorenz also discovered the Lorenz Attractor, an area that pulls points towards itself. He did so during a 3D weather simulation.



The Lorenz Attractor

Chaos Theory

Chaos theory describes complex motion and the dynamics of sensitive systems. Chaotic systems are mathematically deterministic but nearly impossible to predict. Chaos is more evident in long-term systems than in short-term systems. Behavior in chaotic systems is aperiodic, meaning that no variable describing the state of the system undergoes a regular repetition of values. A chaotic system can actually evolve in a way that appears to be smooth and ordered, however. Chaos refers to the issue of whether or not it is possible to make accurate long-term predictions of any system if the initial conditions are known to an accurate degree.



Chaotic systems, in this case a fractal, can appear to be smooth and ordered.

Initial Conditions

Chaos occurs when a system is very sensitive to initial conditions. Initial conditions are the values of measurements at a given starting time. The phenomenon of chaotic motion was considered a mathematical oddity at the time of its discovery, but now physicists know that it is very widespread and may even be the norm in the universe. The weather is an example of a chaotic system. In order to make long-term weather forecasts it would be necessary to take an infinite number of measurements, which would be impossible to do. Also, because the atmosphere is chaotic, tiny uncertainties would eventually overwhelm any calculations and defeat the accuracy of the forecast. The presence of chaotic systems in nature seems to place a limit on our ability to apply deterministic physical laws to predict motions with any degree of certainty.

Chaos on the Large Scale

One of the most interesting issues in the study of chaotic systems is whether or not the presence of chaos may actually produce ordered structures and patterns on a larger scale. It has been found that the presence of chaos may actually be necessary for larger scale physical patterns, such as mountains and galaxies, to arise. The presence of chaos in physics is what gives the universe its "arrow of time", the irreversible flow from the past to the future. For centuries mathematicians and physicists have overlooked dynamical systems as being random and unpredictable. The only systems that could be understood in the past were those that were believed to be linear, but in actuality, we do not live in a linear world at all. In this world linearity is incredibly scarce. The reason physicists didn’t know about and study chaos earlier is because the computer is our "telescope" when studying chaos, and they didn’t have computers or anything that could carry out extremely complex calculations in minimal time. Now, thanks to computers, we understand chaos a little bit more each and every day.

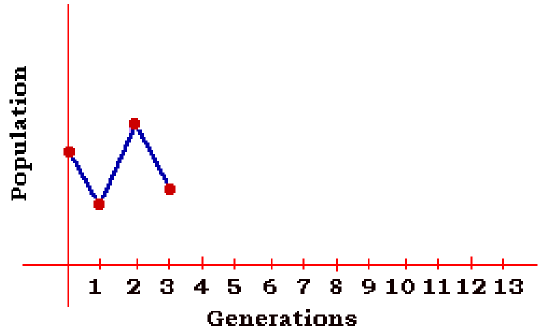
Instability

The definition of instability is a special kind of behavior in time found in certain physical systems. It is impossible to measure to infinite precision, but until the time of Poincaré, the assumption was that if you could shrink the uncertainty in the initial conditions then any imprecision in the prediction would shrink in the same way. In reality, a tiny imprecision in the initial conditions will grow at an enormous rate. Two nearly indistinguishable sets of initial conditions for the same system will result in two final situations that differ greatly from each other. This extreme sensitivity to initial conditions is called chaos. Equilibrium is very rare, and the more complex a system is, there are more disturbances that can threaten stability, but conditions must be right to have an upheaval.

Chaotic systems are instable

Chaos in the Real World

In the real world, there are three very good examples of instability: disease, political unrest, and family and community dysfunction. Disease is unstable because at any moment there could be an outbreak of some deadly disease for which there is no cure. This would cause terror and chaos. Political unrest is very unstable because people can revolt, throw over the government and create a vast war. A war is another type of a chaotic system. Family and community dysfunction is also unstable because if you have a very tiny problem with a few people or a huge problem with many people, the outcome will be huge with many people involved and many people's lives in ruin. Chaos is also found in systems as complex as electric circuits, measles outbreaks, lasers, clashing gears, heart rhythms, electrical brain activity, circadian rhythms, fluids, animal populations, and chemical reactions, and in systems as simple as the pendulum. It also has been thought possibly to occur in the stock market.



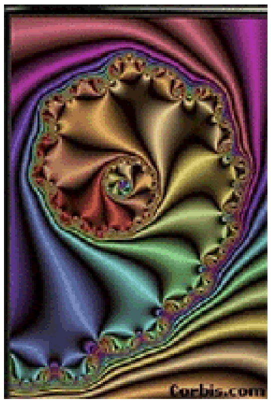
Populations are chaotic, constantly fluctuating, and their graphs can turn out to resemble fractals.

Complexity

Complexity can occur in natural and man-made systems, as well as in social structures and human beings. Complex dynamical systems may be very large or very small, and in some complex systems, large and small components live cooperatively. A complex system is neither completely deterministic nor completely random and it exhibits both characteristics. The causes and effects of the events that a complex system experiences are not proportional to each other. The different parts of complex systems are linked and affect one another in a synergistic manner. There is positive and negative feedback in a complex system. The level of complexity depends on the character of the system, its environment, and the nature of the interactions between them. Complexity can also be called the "edge of chaos". When a complex dynamical chaotic system because unstable, an attractor (such as those ones the Lorenz invented) draws the stress and the system splits. This is called bifurication. The edge of chaos is the stage when the system could carry out the most complex computations. In daily life we see complexity in traffic flow, weather changes, population changes, organizational behavior, shifts in public opinion, urban development, and epidemics.

Fractals

Fractals are geometric shapes that are very complex and infinitely detailed. You can zoom in on a section and it will have just as much detail as the whole fractal. They are recursively defined and small sections of them are similar to large ones. One way to think of fractals for a function f(x) is to consider x, f(x), f(f(x)), f(f(f(x))), f(f(f(f(x)))), etc. Fractals are related to chaos because they are complex systems that have definite properties.



Fractals are recursively defined and infinitely detailed

Benoit Mandelbrot

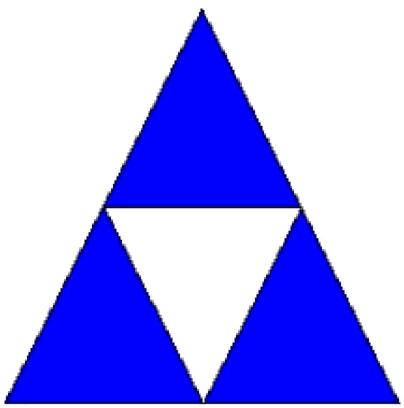
Benoit Mandelbrot was a Poland-born French mathematician who greatly advanced fractals. When he was young, his father showed him the Julia set of fractals; he was not greatly interested in fractals at the time but in the 1970's, he became interested again and he greatly improved upon them, laying out the foundation for fractal geometry. He also advanced fractals by showing that fractals cannot be treated as whole-number dimensions; they must instead have fractional dimensions. Benoit Mandelbrot believed that fractals were found nearly everywhere in nature, at places such as coastlines, mountains, clouds, aggregates, and galaxy clusters. He currently works at IBM's Watson

Benoit Mandelbrot

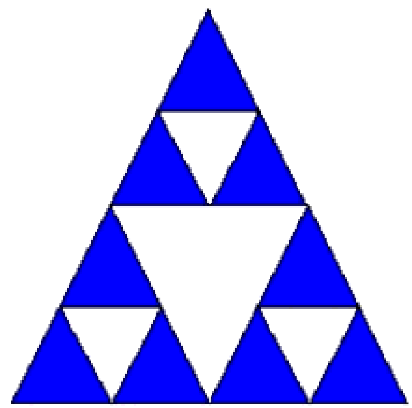
Research Center and is a professor at Yale University. He has been awarded the Barnard Medal for Meritorious Service to Science, the Franklin Medal, the Alexander von Humboldt Prize, the Nevada Medal, and the Steinmetz Medal for his works.

Sierpinski's Triangle

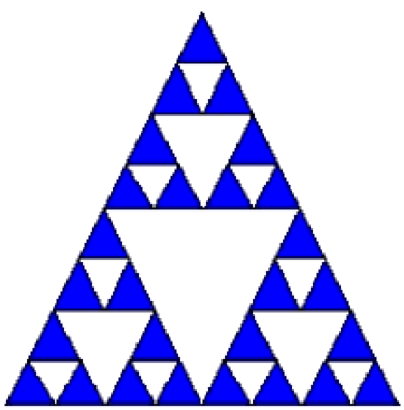
Sierpinski's Triangle is a great example of a fractal, and one of the simplest ones. It is recursively defined and thus has infinite detail. It starts as a triangle and every new iteration of it creates a triangle with the midpoints of the other triangles of it. Sierpinski's Triangle has an infinite number of triangles in it.



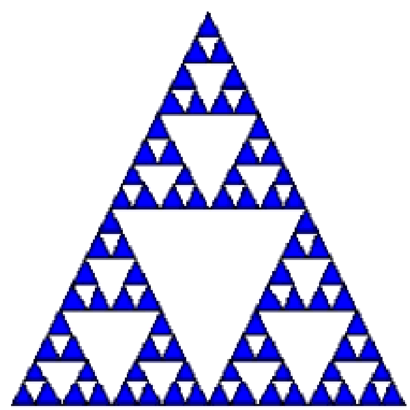
The first recursion of Sierpinski's Triangle



The second recursion of Sierpinski's Triangle



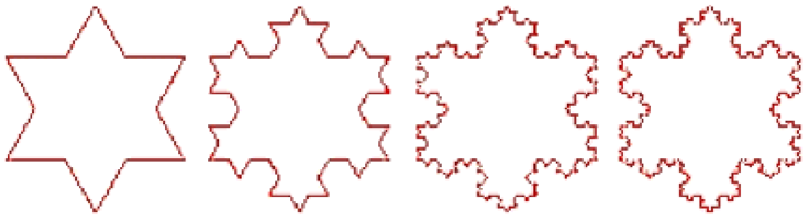
The third recursion of Sierpinski's Triangle



The fourth recursion of Sierpinski's Triangle

Koch Snowflake

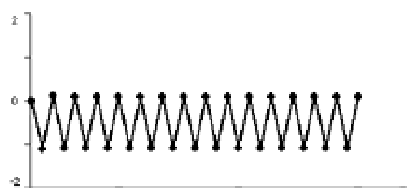
The Koch Snowflake is another good example of a fractal. It starts as a triangle and adds on triangles to its trisection points that point outward for all infinity. This causes it to look like a snowflake after a few iterations.



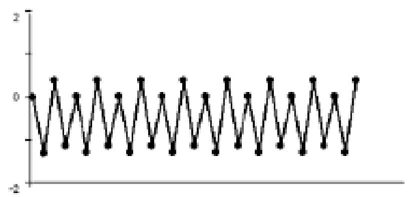
The Koch Snowflake

Mandelbrot Set

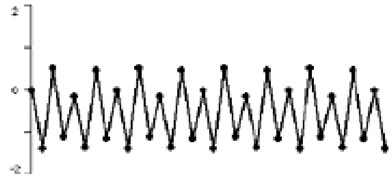
The Mandelbrot fractal set is the simplest nonlinear function, as it is defined recursively as f(x)=x^2+c. After plugging f(x) into x several times, the set is equal to all of the expressions that are generated. The plots below are a time series of the set, meaning that they are the plots for a specific c. They help to demonstrate the theory of chaos, as when c is -1.1, -1.3, and -1.38 it can be expressed as a normal, mathematical function, whereas for c = -1.9 you can't. In other words, when c is -1.1, -1.3, and -1.38 the function is deterministic, whereas when c = -1.9 the function is chaotic.



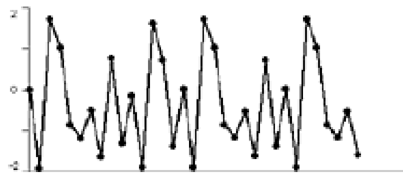
Time Series for c = -1.1



Time Series for c = -1.3



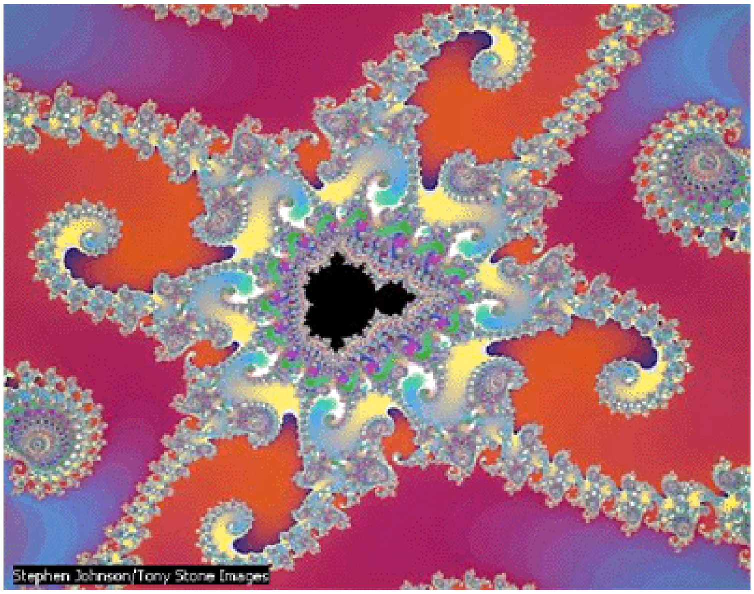
Time Series for c = -1.38



Time Series for c = -1.9

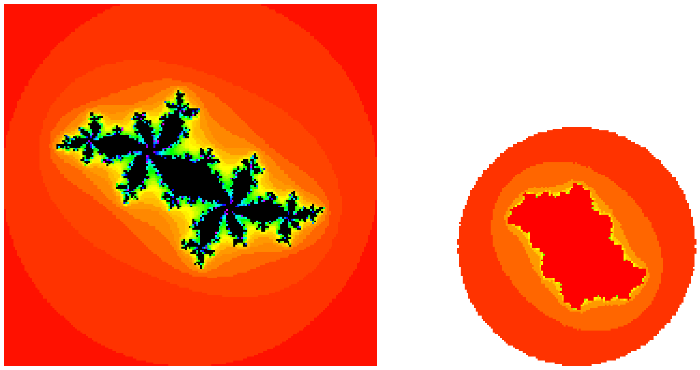
Complex Fractals

When changing the values for the Mandelbrot fractal set from lines to geometric shapes that depend on the various values, a much more complicated picture arises. You can also change the type of system that you use when graphing the fractals and the types of sets that you use in order to generate increasingly complex fractals. The following fractals are very mathematically complex:



Mandelbrot Set Fractal

Julia Set Fractal



Julia Set Fractals

Chaos Theory and Fractal Links

- **Chaos Theory and Fractal Phenomena**: An interesting essay by Manus J. Donahue III that has been cited in *The New York Times*. Contains interesting information on what chaos theory and fractals are and about their history.
- **Chaos Theory Overview**: A very thorough description about the history of chaos, instability, the strange attractor, phase transition, deep chaos, and self organization.
- **Fractal Geometry of the Mandelbrot Set**: A mathematical description about how fractals, particularly the Mandelbrot and Julia Sets, are generated.
- **What is Chaos**: An introductory overview about chaos concerning what determinism, initial conditions, uncertainty, dynamic instabilities, and some manifestations in nature of chaos are.

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- "Chaos Theory History." <http://www.wfu.edu/~petrejh4/HISTORYchaos.htm> (7 June 2000).
- "Fractal Geometry of the Mandelbrot Set." <http://math.bu.edu/DYSYS/FRACGEOM/FRACGEOM.html> (7 June 2000).
- Microsoft Encarta Deluxe.
- "What is Chaos?" <http://order.ph.utexas.edu/chaos/> (7 June 2000).

Web design by Jonathan Mendelson
Text by Jonathan Mendelson and Elana Blumenthal
URL: http://www.mathjmendl.org/chaos/index.html

The Chaos Theory

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An Introduction to Mathematical Chaos Theory and Fractal Geometry

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ATTENTION:

I wrote this explanatory paper on chaos theory about five years ago. The number of interested visitors I get constantly amazes me, some 100,000 in just the last two years alone! In the wake of an economic as well as a technological revolution, it's comforting to see the number of intelligent people out there interested in understanding (at some basic level) a pure science such as chaos theory. Due to the incredible success of this essay (numerous online awards as well as a citation in the *New York Times*), I have taken it upon myself to write a book that deals with what I feel to be some of the most interesting breakthroughs in theoretical physics to date. In addition to a chapter on chaos theory (approximately four times as long as this introductory essay and with more interesting examples), I have also included chapters on relativity, quantum mechanics, string theory and the universe, and even a section on the applications of such theories and how they're shaping the future of both science and society. The book is a combination of philosophy and theoretical physics (as I believe the two to be related), skipping over the burdensome mathematics of the theories (as most people have better things to do :) in favor of raw explanatory emphasis, interesting anecdotes and helpful examples. The book, entitled *News of the Universe: Life at the Edge of Discovery* should be available by the end of the summer of 2002. I'm trying my best to keep the length as well as the cost relatively low, so as to not scare off any potential readers. If you would like more information about the book, then please e-mail me and I'll be happy to oblige. Furthermore, if you would like to request a copy of the book now, then I would be happy, as the author, to personally sign the copy. Thanks a lot - I look forward to hearing from you. In the meantime, enjoy the paper.

The following essay was compiled by me, Manus J. Donahue III (physics and philosophy double-major from Duke University). It has been unofficially published in four different countries, has been cited in The New York Times and has been awarded technology site of the day by TechSightings.com. Please cite this page as a reference if you use any of the material on this page in essays, documents, or presentaitons. Also, you may e-mail me at mjd@duke.edu if you have any questions, and I'll try to get back with you as soon as possible.

Because I compiled this essay for myself and the enjoyment of others, and because I am presenting it completely free, I am not responsible for any copyright violations or anything like that. Some of the pictures that are included in this essay (although almost universally common) were taken from other Web pages. If you are a high school/middle school student who has to do a report on chaos theory and you print this essay off and turn it in, you will be violating not only the work of myself, but the various other people who unknowingly may have contributed to this site. Don't do that - these people deserve credit for their work! Use this paper merely as a ?jumping off point? for your own research, and then write a paper that is even better - and publish it. I wrote this essay because I was always fascinated by chaos theory and non-linear math and I could never find explanative essays aimed at the ?average person.? Hopefully this one is.

"Physicists like to think that all you have to do is say, these are the conditions, now what happens next?" *-Richard P. Feynman*

The world of mathematics has been confined to the linear world for centuries. That is to say, mathematicians and physicists have overlooked dynamical systems as random and unpredictable. The only systems that could be understood in the past were those that were believed to be linear, that is to say, systems that follow predictable patterns and arrangements. Linear equations, linear functions, linear algebra, linear programming, and linear accelerators are all areas that have been understood and mastered by the human race. However, the problem arises that we humans do not live in an even remotely linear world; in fact, our world should indeed be categorized as nonlinear; hence, proportion and linearity is scarce. How may one go about pursuing and understanding a nonlinear system in a world that is confined to the easy, logical linearity of everything? This is the question that scientists and mathematicians became burdened with in the 19th Century; hence, a new science and mathematics was derived: chaos theory.

The very name "chaos theory" seems to contradict reason, in fact it seems somewhat of an oxymoron. The name "chaos theory" leads the reader to believe that mathematicians have discovered some new and definitive knowledge about utterly random and incomprehensible phenomena; however, this is not entirely the case. The acceptable definition of chaos theory states, chaos theory is the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems. A dynamical system may be defined to be a simplified model for the time-varying behavior of an actual system, and aperiodic behavior is simply the behavior that occurs when no variable describing the state of the system undergoes a regular repetition of values. Aperiodic behavior never repeats and it continues to manifest the effects of any small perturbation; hence, any prediction of a future state in a given system that is aperiodic is impossible. Assessing the idea of aperiodic behavior to a relevant example, one may look at human history. History is indeed aperiodic since broad patterns in the rise and fall of civilizations may be sketched;

however, no events ever repeat exactly. What is so incredible about chaos theory is that unstable aperiodic behavior can be found in mathematically simply systems. These very simple mathematical systems display behavior so complex and unpredictable that it is acceptable to merit their descriptions as random.

An interesting question arises from many skeptics concerning why chaos has just recently been noticed. If chaotic systems are so mandatory to our every day life, how come mathematicians have not studied chaos theory earlier? The answer can be given in one word: computers. The calculations involved in studying chaos are repetitive, boring and number in the millions. No human is stupid enough to endure the boredom; however, a computer is always up to the challenge. Computers have always been known for their excellence at mindless repetition; hence, the computer is our telescope when studying chaos. For, without a doubt, one cannot really explore chaos without a computer.

Before advancing into the more precocious and advanced areas of chaos, it is necessary to touch on the basic principle that adequately describes chaos theory, the Butterfly Effect. The Butterfly Effect was vaguely understood centuries ago and is still satisfactorily portrayed in folklore:

"For want of a nail, the shoe was lost;
For want of a shoe, the horse was lost;
For want of a horse, the rider was lost;
For want of a rider, a message was lost;
For want of a message the battle was lost;
For want of a battle, the kingdom was lost!"

Small variations in initial conditions result in huge, dynamic transformations in concluding events. That is to say that there was no nail, and, therefore, the kingdom was lost. The graphs of what seem to be identical, dynamic systems appear to diverge as time goes on until all resemblance disappears.

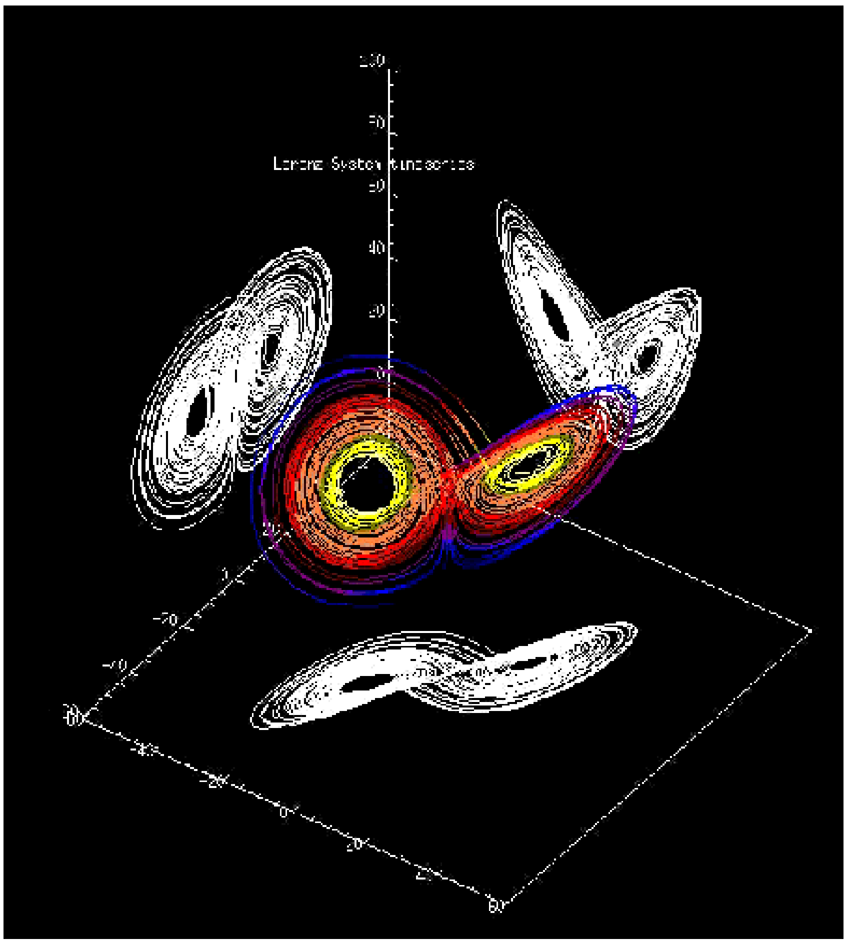
Perhaps the most identifiable symbol linked with the Butterfly Effect is the famed Lorenz Attractor. Edward Lorenz, a curious meteorologist, was looking for a way to model the action of the chaotic behavior of a gaseous system. Hence, he took a few equations from the physics field of fluid dynamics, simplified them, and got the following three-dimensional system:

$$dx/dt=delta*(y-x)$$
$$dy/dt=r*x-y-x*z$$
$$dz/dt=x*y-b*z$$

Delta represents the "Prandtl number," the ratio of the fluid viscosity of a substance to its thermal conductivity; however, one does not have to know the exact value of this constant; hence, Lorenz simply used 10. The variable "r" represents the difference in temperature between the top and bottom of the gaseous system. The variable "b" is the width to height ratio of the box which is being used to hold the gas in the gaseous system. Lorenz used 8/3 for this variable. The resultant x of the equation represents the rate of rotation of the cylinder, "y" represents the difference in temperature at opposite sides of the cylinder, and the variable "z" represents the deviation of the system from a linear, vertical graphed line representing temperature. If one were to plot the three differential equations on a three-dimensional plane, using the help of a computer of course, no geometric structure or even complex curve would appear; instead, a weaving object known as the Lorenz Attractor appears. Because the system never exactly repeats itself, the trajectory never intersects itself. Instead it loops around forever. I have included a computer animated Lorenz Attractor which is quite similar to the production of Lorenz himself. The following Lorenz Attractor was generated by running data through a 4th-order Runge-Kutta fixed-timestep integrator with a step of .0001, printing every 100th data point. It ran for 100 seconds, and only took the last 4096 points. The original parameters were a =16, r =45, and b = 4 for the following equations (similar to the original Lorenz equations):

$$x' = a(y - x)$$
$$y' = rx - y - xz$$
$$z' = xy - bz$$

The initial position of the projectory was (8,8,14). When the points were generated and graphed, the Lorenz Attractor was produced in 3-D:



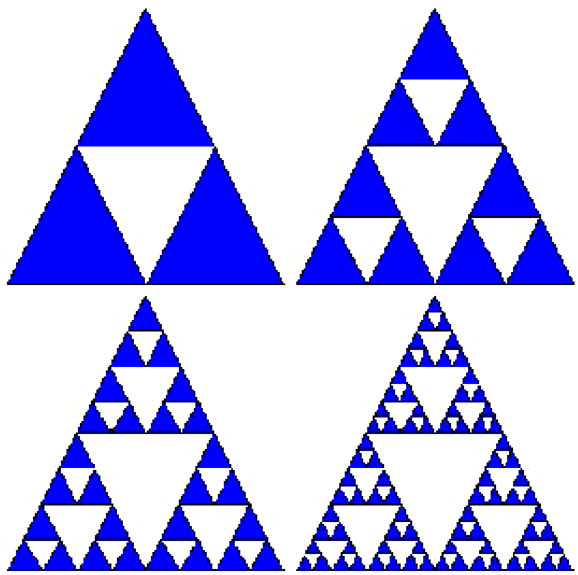
The attractor will continue weaving back and forth between the two wings, its motion seemingly random, its very action mirroring the chaos which drives the process. Lorenz had obviously made an immense breakthrough in not only chaos theory, but life. Lorenz had proved that complex, dynamical systems show order, but they never repeat. Since our world is classified as a dynamical, complex system, our lives, our weather, and our experiences will never repeat; however, they should form patterns.

Lorenz, not quite convinced with his results, did a follow-up experiment in order to support his previous conclusions. Lorenz established an experiment that was quite simple; it is known today as the Lorenzian Waterwheel. Lorenz took a waterwheel; it had about eight buckets spaced evenly around its rim with a small hole at the bottom of each . The buckets were mounted on swivels, similar to Ferris-wheel seats, so that the buckets would always point upwards. The entire system was placed under a waterspout. A slow, constant stream of water was propelled from the waterspout; hence, the waterwheel began to spin at a fairly constant rate. Lorenz decided to increase the flow of water, and, as predicted in his Lorenz Attractor, an interesting phenomena arose. The increased velocity of the water resulted in a chaotic motion for the waterwheel. The waterwheel would revolve in one direction as before, but then it would suddenly jerk about and revolve in the opposite direction. The filling and emptying of the buckets was no longer synchronized; the system was now chaotic. Lorenz observed his mysterious waterwheel for hours, and, no matter how long he recorded the positions and contents of the buckets, there was never and instance where the waterwheel was in the same position twice. The waterwheel would continue on in chaotic behavior without ever repeating any of its previous conditions. A graph of the waterwheel would resemble the Lorenz Attractor.

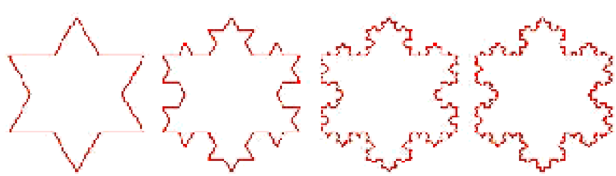
Now it may be accepted from Lorenz and his comrades that our world is indeed linked with an eery form of chaos. Chaos and randomness are no longer ideas of a hypothetical world; they are quite realistic here in the status quo. A basis for chaos is established in the Butterfly Effect, the Lorenz Attractor, and the Lorenz Waterwheel; therefore, there must be an immense world of chaos beyond the rudimentary fundamentals. This new form mentioned is highly complex, repetitive, and replete with intrigue.

"I coined fractal from the Latin adjective fractus. The corresponding Latin verb frangere means "to break": to create irregular fragments. It is therefore sensible-and how appropriate for our needs!-that, in addition to "fragmented", fractus should also mean "irregular," both meanings being preserved in fragment."
-Benoit Mandelbrot

The extending and folding of chaotic systems give strange attractors, such as the Lorenz Attractor, the distinguishing characteristic of a nonintegral dimension. This nonintegral dimension is most commonly referred to as a fractal dimension. Fractals appear to be more popular in the status quo for their aesthetic nature than they are for their mathematics. Everyone who has seen a fractal has admired the beauty of a colorful, fascinating image, but what is the formula that makes up this glitzy image? The classical Euclidean geometry that one learns in school is quite different than the fractal geometry mainly because fractal geometry concerns nonlinear, nonintegral systems while Euclidean geometry is mainly oriented around linear, integral systems. Hence, Euclidean geometry is a description of lines, ellipses, circles, etc. However, fractal geometry is a description of algorithms. There are two basic properties that constitute a fractal. First, is self-similarity, which is to say that most magnified images of fractals are essentially indistinguishable from the unmagnified version. A fractal shape will look almost, or even exactly, the same no matter what size it is viewed at. This repetitive pattern gives fractals their aesthetic nature. Second, as mentioned earlier, fractals have non-integer dimensions. This means that they are entirely different from the graphs of lines and conic sections that we have learned about in fundamental Euclidean geometry classes. By taking the midpoints of each side of an equilateral triangle and connecting them together, one gets an interesting fractal known as the Sierpenski Triangle. The iterations are repeated an infinite number or times and eventually a very simple fractal arises:



In addition to the famous Sierpenski Triangle, the Koch Snowflake is also a well noted, simple fractal image. To construct a Koch Snowflake, begin with a triangle with sides of length 1. At the middle of each side, add a new triangle one-third the size; and repeat this process for an infinite amount of iterations. The length of the boundary is 3 X 4/3 X 4/3 X 4/3...-infinity. However, the area remains less than the area of a circle drawn around the original triangle. What this means is that an infinitely long line surrounds a finite area. The end construction of a Koch Snowflake resembles the coastline of a shore.

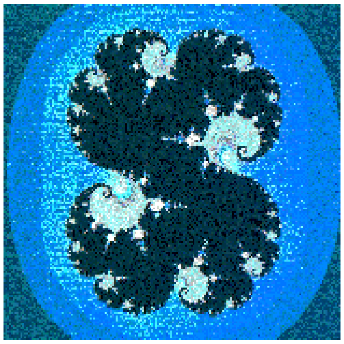
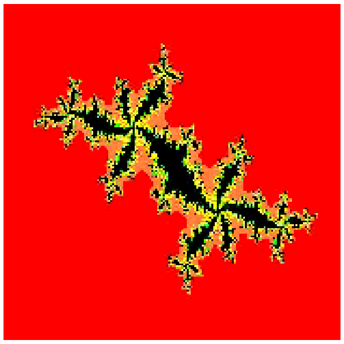


The two fundamental fractals that I have included provided a basis for much more complex, elaborate fractals. Two of the leading reasearchers in the field of fractals were Gaston Maurice Julia and Benoit Mandelbrot. Their discoveries and breakthroughs will be discussed next.

On February 3rd, 1893, Gaston Maurice Julia was born in Sidi Bel Abbes, Algeria. Julia was injured while fighting in World War I and was forced to wear a leather strap across his face for the rest of his life in order to protect and cover his injury. he spent a large majority of his life in hospitals; therefore, a lot of his mathematical research took place in the hospital. At the age of 25, Julia published a 199 page masterpiece entitled "Memoire sur l'iteration des fonctions." The paper dealt with the iteration of a rational function. With the publication of this paper came his claim to fame. Julia spent his life studying the iteration of polynomials and rational functions. If f(x) is a function, various behaviors arise when "f" is iterated or repeated. If one were to start with a particular value for x, say x=a, then the following would result:

a, f(a), f(f(a)), f(f(f(a))), etc.

Repeatedly applying "f" to "a" yields arbitrarily large values. Hence, the set of numbers is partitioned into two parts, and the Julia set associated to "f" is the boundary between the two sets. The filled Julia set includes those numbers x=a for which the iterates of "f" applied to "a" remain bounded. The following fractals belong to the Julia set.

Julia became famous around the 1920's; however, upon his demise, he was essentially forgotten. It was not until 1970 that the work of Gaston Maurice Julia was revived and popularized by Polish born Benoit Mandelbrot.

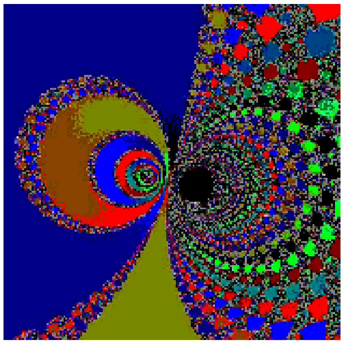
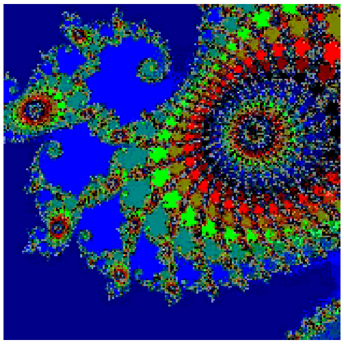
"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line." *-Benoit Mandelbrot*

Benoit Mandelbrot was born in Poland in 1924. When he was 12 his family emigrated to France and his uncle, Szolem Mandelbrot, took responsibility for his education. It is said that Mandelbrot was not very successful in his schooling; in fact, he may have never learned his multiplication tables. When Benoit was 21, his uncle showed him Julia's important 1918 paper concerning fractals. Benoit was not overly impressed with Julia's work, and it was not until 1977 that Benoit became interested in Julia's discoveries. Eventually, with the aid of computer graphics, Mandelbrot was able to show how Julia's work was a source of some of the most beautiful fractals known today. The Mandelbrot set is made up of connected points in the complex plane. The simple equation that is the basis of the Mandelbrot set is included below.

changing number + fixed number = Result

In order to calculate points for a Mandelbrot fractal, start with one of the numbers on the complex plane and put its value in the "Fixed Number" slot of the equation. In the

"Changing number" slot, start with zero. Next, calculate the equation. Take the number obtained as the result and plug it into the "Changing number" slot. Now, repeat (iterate) this operation an infinite number or times. When iterative equations are applied to points in a certain region of the complex plane, a fractal from the Mandelbrot set results. A few fractals from the Mandelbrot set are included below.

Benoit Mandelbrot currently works at IBM’s Watson Research Center. In addition, he is a Professor of the Practice of Mathematics at Harvard University. He has been awarded the Barnard Medal for Meritorious Service to Science, the Franklin Medal, the Alexander von Humboldt Prize, the Nevada Medal, and the Steinmetz Medal. His work with fractals has truly influenced our world immensely.

It is now established that fractals are quite real and incredible; however, what do these newly discovered objects have to do with real life? Is there a purpose behind these fascinating images? The answer is a somewhat surprising yes. Homer Smith, a computer engineer of Art Matrix, once said, "If you like fractals, it is because you are made of them. If you can’t stand fractals, it’s because you can’t stand yourself." Fractals make up a large part of the biological world. Clouds, arteries, veins, nerves, parotid gland ducts, and the bronchial tree all show some type of fractal organization. In addition, fractals can be found in regional distribution of pulmonary blood flow, pulmonary alveolar structure, regional myocardial blood flow heterogeneity, surfaces of proteins, mammographic parenchymal pattern as a risk for breast cancer, and in the distribution of arthropod body lengths. Understanding and mastering the concepts that govern fractals will undoubtedly lead to breakthroughs in the area of biological understanding. Fractals are one of the most interesting branches of chaos theory, and they are beginning to become ever more key in the world of biology and medicine.

George Cantor, a nineteenth century mathematician, became fascinated by the infinite number of points on a line segment. Cantor began to wonder what would happen when and infinite number of line segments were removed from an initial line interval. Cantor devised an example which portrayed classical fractals made by iteratively taking away something. His operation created a "dust" of points; hence, the name Cantor Dust. In order to understand Cantor Dust, start with a line; remove the middle third; then remove the middle third of the remaining segments; and so on. The operation is shown below.

The Cantor set is simply the dust of points that remain. The number of these points are infinite, but their total length is zero. Mandelbrot saw the Cantor set as a model for the occurerence of errors in an electronic transmission line. Engineers saw periods of errorless transmission, mixed with periods when errors would come in gusts. When these gusts of errors were analyzed, it was determined that they contained error-free periods within them. As the transmissions were analyzed to smaller and smaller degrees, it was determined that such dusts, as in the Cantor Dust, were indispensable in modeling intermittency.

"It's an experience like no other experience I can describe, the best thing that can happen to a scientist, realizing that something that's happened in his or her mind exactly corresponds to something that happens in nature. It's startling every time it occurs. One is surprised that a construct of one's own mind can actually be realized in the honest-to-goodness world out there. A great shock, and a great, great joy."
-Leo Kadanoff

The fractals and iterations are fun to look at; the Cantor Dust and Koch Snowflakes are fun to think about, but what breakthroughs can be made in terms of discovery? Is chaos theory anything more than a new way of thinking? The future of chaos theory is unpredictable, but if a breakthrough is made, it will be huge. However, miniature discoveries have been made in the field of chaos within the past century or so, and, as expected, they are mind boggling.

The first consumer product to exploit chaos theory was produced in 1993 by Goldstar Co. in the form of a revolutionary washing machine. A chaotic washing machine? The washing machine is based on the principle that there are identifiable and predictable movements in nonlinear systems. The new washing machine was designed to produce cleaner and less tangled clothes. The key to the chaotic cleaning process can be found in a small pulsator that rises and falls randomly as the main pulsator rotates. The new machine was surprisingly successful. However, Daewoo, a competitor of Goldstar claims that they first started commercializing chaos theory in their "bubble machine" which was released in 1990. The "bubble machine" was the first to use the revolutionary "fuzzy logic circuits." These circuits are capable of making choices between zero and one, and between true and false. Hence, the "fuzzy logic circuits" are responsible for controlling the amount of bubbles, the turbulence of the machine, and even the wobble of the machine. Indeed, chaos theory is very much a factor in today's consumer world market.

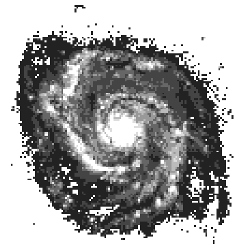
The stock markets are said to be nonlinear, dynamic systems. Chaos theory is the mathematics of studying such nonlinear, dynamic systems. Does this mean that chaoticians can predict when stocks will rise and fall? Not quite; however, chaoticians have determined that the market prices are highly random, but with a trend. The stock market is accepted as a self-similar system in the sense that the individual parts are related to the whole. Another self-similar system in the area of mathematics are fractals. Could the stock market be associated with a fractal? Why not? In the market price action, if one looks at the market monthly, weekly, daily, and intra day bar charts, the structure has a similar appearance. However, just like a fractal, the stock market has sensitive dependence on initial conditions. This factor is what makes dynamic market systems so difficult to predict. Because we cannot accurately describe the current situation with the detail necessary, we cannot accurately predict the state of the system at a future time. Stock market success can be predicted by chaoticians. Short-term investing, such as intra day exchanges are a waste of time. Short-term traders will fail over time due to nothing more than the cost of trading. However, over time, long-term price action is not random. Traders can succeed trading from daily or weekly charts if they follow the trends. A system can be random in the short-term and deterministic in the long term.

Perhaps even more important than stock market chaos and predictability is solar system chaos. Astronomers and cosmologists have known for quite some time that the solar system does not "run with the precision of a Swiss watch." Inabilities occur in the motions of Saturn’s moon Hyperion, gaps in the asteroid belt between Mars and Jupiter, and in the orbit of the planets themselves. For centuries astronomers tried to compare the solar system to a gigantic clock around the sun; however, they found that their equations never actually predicted the real planets’ movement. It is easy to understand how two bodies will revolve around a common center of gravity. However, what happens when a third, fourth, fifth or infinite number of gravitational attractions are introduced? The vectors become infinite and the system becomes chaotic. This prevents a definitive analytical solution to the equations of motion. Even with the advanced computers that we have today, the long term calculations are far too lengthy. Stephen Hawking once said, "If we find the answer to that (the universe), it would be the ultimate triumph of human reason-for then we would know the mind of God.

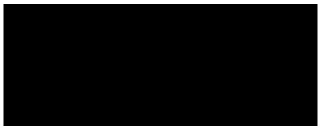
The applications of chaos theory are infinite; seemingly random systems produce patterns of spooky understandable irregularity. From the Mandelbrot set to turbulence to feedback and strange attractors; chaos appears to be everywhere. Breakthroughs have been made in the past in the area chaos theory, and, in order to achieve any more colossal accomplishments in the future, they must continue to be made. Understanding chaos is understanding life as we know it.

However, if we do discover a complete theory, it should in time be understandable in broad principle by everyone, not just a few scientists. Then we shall all, philosophers, scientists, and just ordinary people, be able to take part in the discussion of the question of why it is that we and the universe exist. -*Stephen Hawking*

π CHAOS THEORY



" That is the truth of our world, Max. It can't be easily summed up with math..."
- Sol in π.



K.A.O.S.
A cocktail-party primer on Chaos with access to deeper areas of this dangerous field.

Bulls, Bears and Butterflies
The Stock Market as a chaotic system.

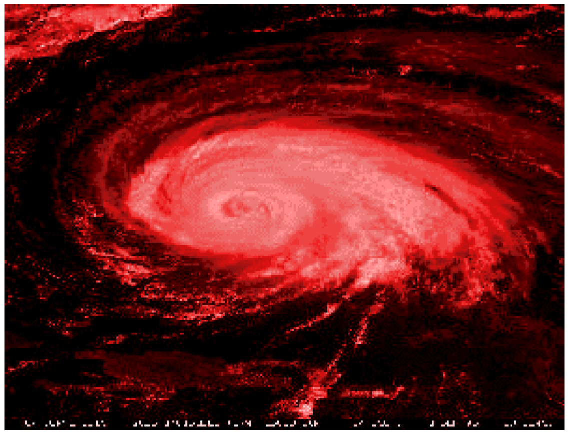
Chaos Theory can be generally defined as the study of forever-changing complex systems. Discovered by a meteorologist in 1960, chaos theory contends that complex and unpredictable results will occur in systems that are sensitive to small changes in their initial conditions. The most common example of this, known as the "Butterfly Effect," states that the flapping of a butterfly's wings in China could cause tiny atmospheric changes which over a period of time could effect weather patterns in New York.

Although chaotic systems appear to be random, they are not. Beneath the random behavior patterns emerge, suggesting, if not always revealing, order. Recognizing that the stock market is a non-linear, dynamic, chaotic system π's Max Cohen (Sean Gullette) applies the principles of Chaos Theory in order to determine the pattern behind apparent random nature of market prices.

Apart from the stock market, Chaos Theory can be used to model other highly complex systems, including everything from population growth to epidemics to arrhythmic heart palpitations. When applying chaos theory, it is revealed that even something as seemingly random as a dripping faucet has an order behind it.

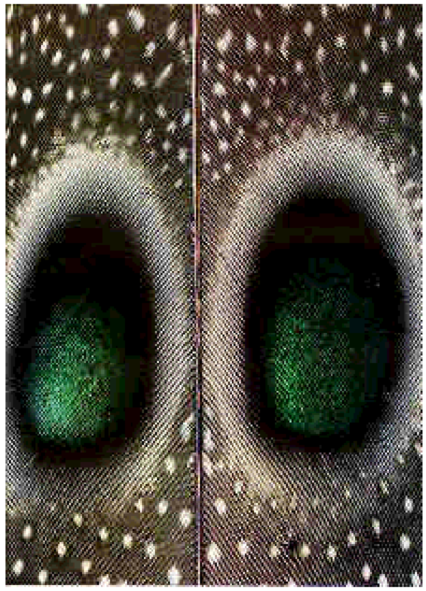
CHAOS COMPLEXITY THEORY

CHAOS WITHOUT THE MATH



Below are links to chaos theory and weblinks to fractal pictures. More important links on the new site above. The new site above gives the reader an idea of what Chaos is without using the math involved. Chaos/Complexity theory possesses explanations that haunt many academic disciplines including science, philosophy, sociology and psychology as to why they seem to overlap. Research by this author will suggest that Complexity Theory offers some very distinctive qualities to Theology also.

- **COMPLEXITY THEORY What is it and how can I use it?**
- **Nonlinearity/Complexity : links galore!**
- **Chaos Network: wonderful general chaos site**
- **Fractal Pictures/Animations: go here for graphics**
- **Chaos at Maryland: a comprehensive chaos collection**
- **GeorgiaTech ACL: the Applied Chaos Laboratory.**
- **Contours of the Mind: an impressive *artistic* effort.**
- **CHAOS LINKS--An extensive list of Chaos sites.**



This image is a detail of a feather.
As an example of Chaos in Nature,
it shows the self-organizing side of Complexitiy.

Judy’s Home Page

Fractals and Chaos Theory.

Mathematical Chaos

Fractals and chaos theory have not only had a dramatic impact on mathematics, but also art and computer visualization. Some people might argue that chaos and fractals were around long before the invention of computers, especially considering Poincare's researches, and the fact that Cantor had invented an infinite set, that was essentially a "fractal". However it was not until computers began to be used extensively in numerical research in the 1960's that chaos and fractals would become a fully-fledged area of mathematics to display themselves in a dizzying array of colours and infinite descent that so many people have seen.

So how did it begin? Most mathematical physics is based around the study and solution of Differential Equations. Differential Equations are just like every other sort of equation, in which something on one side has to equal something on the other side, for example

$$y = t^2 + t + c$$

For the above equation we are usually given the values of y and c, and asked to solve the equation for t. Thus we have one side equaling the other side, and we are asked to find certain values of t that makes this true. Differential equations are of exactly the same form, except instead of solving for simple values or functions, we are asked to solve for functions that relate the rate of change of one value to another.

For clarity consider the following example.

$$\frac{dy}{dt} = y$$

The left-hand side of the above equation simple reads as "the rate of change of some function y, with respect to a variable t". For instance we could have y standing for the amount of water in a given volume at a particular time t. Then the above equation would read "the rate of change of water in a given volume at a particular time t, is equal to the amount of water in that volume at time t."

Thus with such an equation we are asked to find y such that its rate of change is equal to its self.

All differential equations not matter how complicated or involved can be related to this simple idea of solving for a function when given a description of its rate of change. Most of these types of equations can not be solved analytically, meaning we can't write down the answer in terms of known functions. For these equations we must resort to numerical methods to obtain a solution. As you can probably see, if we wish to solve for these equations we need to have some initial data that we can put into these equations to obtain the particular solution we are after. For example, in the above differential equation we want to know how a volume of water changes over time, given that its rate of change with time is equal to its self. But for a PARTICULAR problem, we may also be given the information that initially we started of with 10 cubic meters of water, for example. Thus by stating this initial condition we can then go off and solve the equation, determine its solution, thus obtaining a description of how the volume of water changes over time if initially it started off being 10 cubic meters.

This is not only how Chaos theory began, but this is how Modern Science began.

Chaos theory was first discovered by a meteorologist called Edward Lorenz in 1960. Lorenz was studying a particular set of 12 differential equations and their numerical solutions for a given set of initial conditions, which he thought, might be able to predict certain weather conditions. That in its self is not peculiar in the slightest. Mathematicians had been investigating differential equations of all shapes and sizes numerically for a very long time. The only difference between how Lorenz was conducting his investigations and how most people up until that time investigated the equations, was that Lorenz had a small "personal computer" at his disposal, that could turn days and days of horrible arithmetic and algebraic drudgery for even just a small amount of numerical data, into a few hours. His ability to process these equations numerically, far exceeded that of "classical" mathematicians, which is the very reason why Chaos Theory was not discovered earlier than it was. Chaos theory required a computer to be seen.

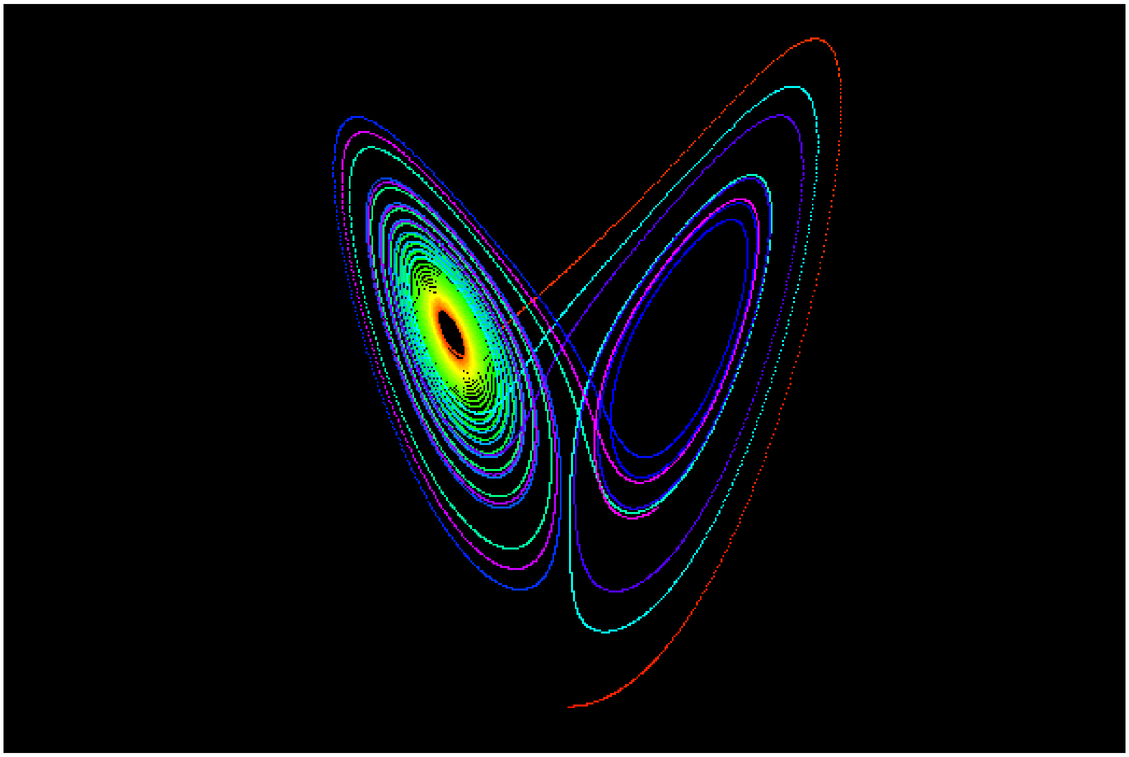
So what was so strange about Lorenz's investigations? Lorenz noticed that if he solved the equations with a particular set of initial conditions, and then went back and did all the calculations again with a slightly varied set of initial conditions, the end results were extraordinarily different. What he had discovered was a sort of "Sensitivity" of these equations to their initial conditions. How did he actually see this variation? Lorenz wished to see a particular sequence of numbers again. He had previously generated a print out of this sequence of numbers that came from solving these differential equations, but wished to generate them again. To save himself some time, instead of starting from scratch and generating all the numbers again, he thought he would use his old sequence and read off a number from the middle of the sequence and let it run from there. This way he would only have to wait half the time to generate the same sequence of numbers he already had. However, his computer stored numbers to six decimal places, and Lorenz, thinking it would not matter, only entered three decimal places. When he came back after a couple of hours, he noticed that the sequence was completely different to that he had previously obtained. What he noticed was a sensitivity to initial conditions, and what is now lovingly termed "The butterfly effect".

The butterfly effect is usually explained by describing a butterfly that flaps its wings, and causes a cyclone on the other side of the world. The essence of the analogy is that tiny variations cause huge effects. In this case it was the small pressure variations from the butterfly's wings that ended up generating huge weather effects miles away. The actual Mathematical description of the butterfly effect, we have just explained. All it is, is that a slight variation of initial conditions inputted into the numerical solution of certain equations, can lead to drastically varied results. This was the beginning of chaos theory, and the embryo of fractals.

Looking at the Order in Chaos

From this starting point Lorenz then started to look at a simpler situation involving only 3 differential equations. What he did was take the equations of Convection and strip them down until they were very simply related, obtaining only 3 equations. It would later be shown that these equations described a water wheel.

This new set of equations also showed this sensitivity to initial conditions, however this time when he graphed the solutions for this system (in what's known as Poincare space, or phase space), he noticed that the output was always situated on two curves.



In what seemed to be an amazing contradiction in terms, there appeared to be some sort of order within the chaotic structure of the numbers generated from these equations.

Similar systems that are sensitive to variations in initial conditions exist in many different areas. Most notably is the classic example in biology and population dynamics. The standard equation of such areas is most simply stated as

$$P_{n+1} = aP_n(1 - P_n)$$

Where

P_n is the previous years population, and is given on a scale from 0 to 1, thus

P_{n+1} is the next years population, and a is a constant called the growth rate.

Such an equation is called a recursive equation, meaning that it will tell you the next lot of values based on the previous values. Most differential equations can be reduced to this type of equation for numerical evaluation.

To see more clearly how this type of equation works lets plug in a few experimental values. Lets say that at year 0 we have a population of .5, and we put a=1 so we can write,

$$P_0 = \frac{1}{2}$$

These are our initial conditions. Then to determine the next years population I will have to use,

$$P_1 = P_0(1 - P_0)$$

which is just from our main formula . Thus substituting we get

$$P_1 = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$$

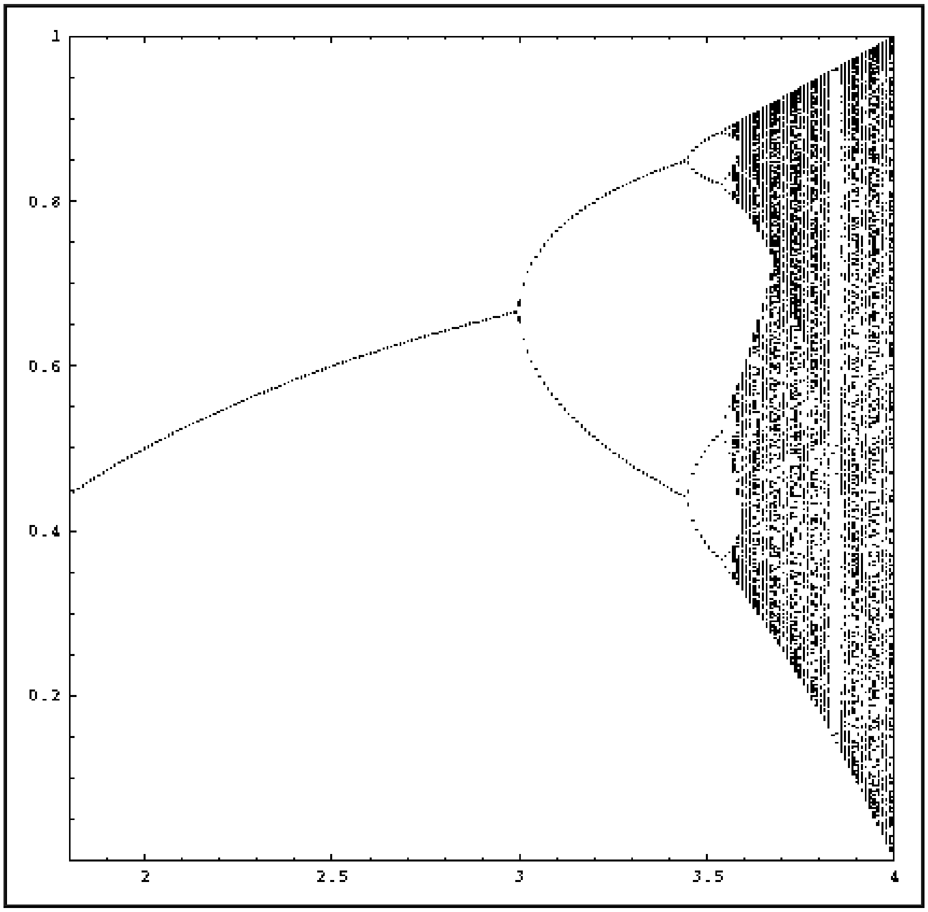
and thus

$$P_2 = \frac{1}{4}(1 - \frac{1}{4}) = \frac{3}{16}$$

and so on and so on. This process by where we determine the next data point from the previous data point is called iteration. The process of iteration is at the very heart of chaos and fractals, it is how the differential equations are solved, and it is how fractals are generated. In the above example we looked at some iterations for a particular growth rate of a=1, but what happens if we change this growth rate?

What happens to the sequence of numbers that are generated by iteration for different rates of growth?

A biologist named Robert May, investigated such questions for this particular equation. He found that if the growth rate was a low number then the iterative process would converge to ONE number. This means that after we perform enough iterative steps, we just keep on getting the same number again and again. Then something very strange happened. He noted that after say about a=3, the equation would not converge to just one number, but converged to two numbers! How can this happen? He noticed that after some iteration the sequence converged to one particular number for a particular year, then the next iteration it was a different number, then the following iteration it returned to the number for the year before. So it was oscillating between two different numbers. More simply stated; if you put one of these numbers into the equation you got the other number. As he further increased the growth rate parameter, the sequence split again (bifurcation.....doubled) and oscillated between four different numbers. The further he increased the growth rate the faster the bifurcation occurred, until it reached a point where it became totally chaotic and the system became unpredictable.



In the above diagram (known as an orbit diagram) we see the values where the iteration converges to a single value. Then further on we see it split (the first bifurcation). A little further on we see it double again, and then again. Then we see the horrible dark mess that is chaos.

If we look a little further on, we see a white strip. In actuality (although you might not be able to make it out) this white strip is where the whole process starts again. We again have single values which intern double, then double again, then once more return to the chaos regime. You will notice there are a few white strips in the diagram. Therefore we are staring at a diagram that actually contains repetitions of ITSELF WITHIN ITSELF. It is this self-similarity that became a very important feature of chaos theory. This should be reminding you of something very familiar by now. In fact it is the one-dimensional version of a very famous object. But more on that later.

There is so much more interesting stuff we could go into about chaos theory, just given the very basics we have covered here. For instance there is the Feigenbaum number, and its relation to the scales of the period doubling areas in the orbit diagram. It is a number that is constant and exists in all iteration schemes such as this. More mysteriously it is an irrational number, and one particular Feigenbaum number pops up in certain phase transitions in physics, and is very nearly equal to

$$\pi + \arctan(e^{-\pi})$$

Astronomers have also witnessed radio signals from space that have peaked from the background noise and are almost exactly equal to the Feigenbaum constant up to approximately 10 decimal places. This is merely the very tip of the iceberg as far as chaos theory goes. There are many more strange properties contained with in its fold, and one of these just happens to be the world of fractals.

The Birth of Fractals

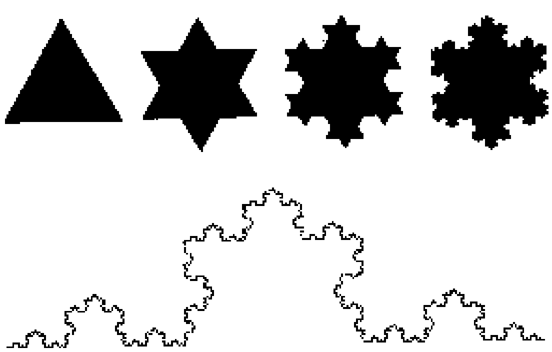
Up to this point we have discussed the main underlying nature of chaos theory, a theory that was born from a natural mistake of omitting a few digits in a numerical computation. Never again were scientists to be so haphazard. However at this stage chaos theory really wasn't "welcomed with open arms" into the scientific community. It was more of a hobby that biologists like to work on now and again and the "lets see what we can come up with" attitude dominated the area. The reason for such semi-rejection from the scientific community was largely based upon the fact that chaos theory at the time, didn't really have a real world problem that it could help solve. It was more of an odd curiosity. And like most of the greatest theories in science, it was at first overlooked.

Poincare had done some previous work in dynamics that touched on chaos theory, and it is to him that we owe the Poincare phase space, where so many chaotic systems are graphed. However he never really got the chance to peer at the numerical intricacies of the subject, simply because he didn't have the computational power to do so at the time.

Cantor on the other hand, I feel to be one of the first persons to glimpse the edge of chaos theory. Cantor was a brilliant mathematician who dedicated his life to studying numbers and the theory of sets. In particular he was the first person to come up with the idea of infinite numbers, and proved the fact that it was indeed possible to have one infinity larger than the other. He too was rejected by the scientific community, however his theories are no considered not only the cornerstone of certain areas of mathematics, but form the building blocks of mathematics itself! And given the nature of his work and the fact that it is probably the most detailed definition of infinity the human race has known it is no surprise that we should find him connected in some way to chaos theory. But how exactly?

The connection comes about from considering self-similarity. Benoit Mandelbrot, an employee of IBM at the time, was considering this very phenomenon of self-similarity, which he had stumbled upon by analyzing cotton prices. He noticed that although the price fluctuations of cotton over the years, never really fitted on a normal distribution, their scale was almost identical. What he saw was that while the daily and monthly pricing of cotton were random, the curves of daily price changes, and monthly price changes were identical. This self similarity with scale lead him to consider other such problems, such as the self-similarity of a coast line. If we look at a coast line on a map, we see bays etc, but we don't see the small bays that are within the bays. We also don't see the small structures that are within those bays and so on.

This idea was more mathematically demonstrated by Helge Von Koch. He constructed a curve now known as the Koch curve. To construct a Koch curve, you start with an equilateral triangle and add another equilateral triangle to the middle third of each side of the original triangle. Then you repeat this ad-infinitum. What you end up with is a structure that if you zoom in on any area of the curve, it looks exactly like the overall curve.



This curve introduces a seemingly delightful paradox. Every time a triangle is added the length of the curve gets larger. The area that the curve bounds however, always remains less than that of a circle draw around the original triangle. So if we keep adding triangles indefinitely, then we have a curve of infinite length surrounding a finite area! What we are looking at is the idea that one infinity is larger than another, and that is exactly what Cantor investigated nearly a hundred years earlier.

To help visualize this new object they had found, mathematicians coined the term fractional dimension. The Koch curve has a fractional dimension of 1.26. Its not too hard to convince your self that this appears to be acceptable. A square that has an ordinary straight line bounding its inner area, has a dimension of two. A Koch curve is more crinkly than a straight line (which has dimension one), so it takes up more "space" than a straight line. So it seems reasonable that the Koch curve should lie somewhere between the two, a straight line and a two dimensional curve. In time any construction that displayed this self-similarity would simply be called a Fractal.

A little later on a researcher by the name of Feigenbaum discovered a truly remarkable property of mathematics. If you took the ratio of any consecutive areas of an orbit diagram, you obtained a constant. As if this fact that for one diagram the ratio of any consecutive areas resulted in the same constant wasn't remarkable enough, the fact that this constant was produced when looking at ratio of areas in ALL other bifurcation systems was surely mind boggling. He had discovered an underlying thread of universality that linked apparently completely different chaotic systems. It was this discovery that provided scientists with some powerful tools to further their research. Chaos theory was no longer a mild curiosity, it was now displaying very unusual and mysterious mathematical phenomenon. From that point on, chaos theory stopped being a hobby, and became a dedicated field of research. Mathematicians who had long since passed away started to get their names attached to this rising field of mathematics. Julia, Koch, Poincare, Dedekind to name but a few, suddenly had a piece of the fractal pie. Their mathematics were aiding researches to delve into chaos theory, and sometimes they found that the mathematics of these older mathematicians were unknowingly at the time, fractals or elements of chaos theory. With all the names and mathematical work emerging, one name came up time and time again. His theories on numbers and sets would prove to be invaluable in the research of chaotic systems. His theories and ideas of infinity, while mocked and considered ridiculous at the time, would help straighten out some of the difficulties with this new theory. Cantor had once again, proved his worth and forever stamped his name as one of the greatest mathematicians the world has ever known.

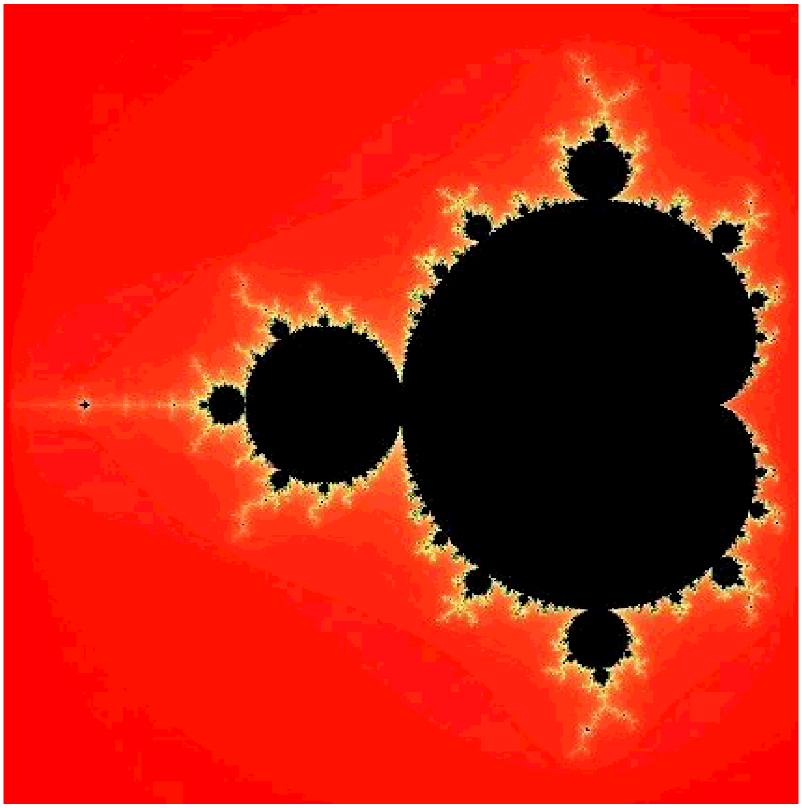
While people had constructed orbit diagrams, and plotted solutions in phase space, to help visualize what was happening, no one had as yet, constructed a diagram of a full two dimensional fractal via a computer. Mandelbrot upon seeing the self-similarity of the cotton prices then began looking at earlier papers by a mathematician named Julia. Julia sets, for their time (1918 or there about) were very strange. A set is just a collection of "things". Julia sets were composed of points that did not go to infinity after applying a certain class of functions repeatedly. A quadratic Julia set is created by considering the following function

$$z_{n+1} = z_n^2 + c$$

Here we see the same sort of equation as for the population dynamics discussed earlier. The Julia set is created by picking a value z, applying the above equation repeatedly (i.e. iterating it), and then seeing if it converges to a number. If it does converge, then we include z in the set, if it doesn't converge, then that particular value of z is not included in the set. There is however one major difference between this scheme and that of the population equation. The values of z belong to the complex plane.

Complex numbers comprise what's known as a real and imaginary part. The real part is your every day real number (any number you can think of is a real number), the imaginary part is any real number multiplied by the square root of minus one. Complex numbers or complex analysis is an entire field of mathematics in its own right, thus writing up a page description of them really wouldn't do them justice. So if you don't know anything about them, just be happy with the fact that they are fundamental to modern analysis and play a huge role in mathematics. When you transfer things over to the complex plane, you usually end up getting very exciting results. Fractals are no exception.

Mandelbrot set about writing a program that would plot a version of the Quadratic Julia set, and thus create the first visualization of a fractal. He succeeded, and was presented with a mind blowing geometry of what is now considered one of the most complex structures in mathematics. It seems very fitting that one of the most complex and famous fractals, is also one of the simplest to write down. This Set later became known as the Mandelbrot Set or Mandelbrot fractal.



The actual Mandelbrot set, that is all the numbers that do not go to infinity, reside right in the middle of the fractal. It is in fact that dull looking black area that no one finds interesting at all. All the groovy looking colours are regions (or values if you like) of the complex plane that if you iterate in the equation, go to infinity. The different colours represent how fast the points not in the set diverge. So all the so-called interesting structure are points that really aren't in the Mandelbrot set.

From here on, the scientific world appeared to go fractal crazy. Chaos theory started to be applied to all sorts of areas of science. From modeling bark structure, to imitating the slight variations in a heart beat there seems to be no end to the application of fractals and chaos theory. It was proposed that the new century science would comprise of three main theories, quantum mechanics, relativity and chaos. It should not be surprising to find that chaos theory seems to model the real world so well. After all we live in a world where irregularities are so common they almost appear with a distinct regularity. Chaos theory also appears to have no bounds as far as scale goes. It has been applied to the motion of molecules while also been used to describe the resonances in gravity of planetary systems.

From considering just how well fractals can describe or imitate natural objects, it is no surprise that a landscape-rendering package such as Bryce has at its core, fractal algorithms. From the actual creation of landscape objects, to the fact that the very rendering engine itself can be used as a fractal, presents infinite possibilities. Infinite possibilities seems rather fitting, given the fact that fractals could be viewed as a glimpse of infinity itself.

Fractals and Chaos Theory.

Mathematical Chaos

Fractals and chaos theory have not only had a dramatic impact on mathematics, but also art and computer visualization. Some people might argue that chaos and fractals were around long before the invention of computers, especially considering Poincare's researches, and the fact that Cantor had invented an infinite set, that was essentially a "fractal". However it was not until computers began to be used extensively in numerical research in the 1960's that chaos and fractals would become a fully-fledged area of mathematics to display themselves in a dizzying array of colours and infinite descent that so many people have seen.

So how did it begin? Most mathematical physics is based around the study and solution of Differential Equations. Differential Equations are just like every other sort of equation, in which something on one side has to equal something on the other side, for example

$$y = t^2 + t + c$$

For the above equation we are usually given the values of y and c, and asked to solve the equation for t. Thus we have one side equaling the other side, and we are asked to find certain values of t that makes this true. Differential equations are of exactly the same form, except instead of solving for simple values or functions, we are asked to solve for functions that relate the rate of change of one value to another.

For clarity consider the following example.

$$\frac{dy}{dt} = y$$

The left-hand side of the above equation simple reads as "the rate of change of some function y, with respect to a variable t". For instance we could have y standing for the amount of water in a given volume at a particular time t. Then the above equation would read "the rate of change of water in a given volume at a particular time t, is equal to the amount of water in that volume at time t."

Thus with such an equation we are asked to find y such that its rate of change is equal to its self.

All differential equations not matter how complicated or involved can be related to this simple idea of solving for a function when given a description of its rate of change. Most of these types of equations can not be solved analytically, meaning we can't write down the answer in terms of known functions. For these equations we must resort to numerical methods to obtain a solution. As you can probably see, if we wish to solve for these equations we need to have some initial data that we can put into these equations to obtain the particular solution we are after. For example, in the above differential equation we want to know how a volume of water changes over time, given that its rate of change with time is equal to its self. But for a PARTICULAR problem, we may also be given the information that initially we started of with 10 cubic meters of water, for example. Thus by stating this initial condition we can then go off and solve the equation, determine its solution, thus obtaining a description of how the volume of water changes over time if initially it started off being 10 cubic meters.

This is not only how Chaos theory began, but this is how Modern Science began.

Chaos theory was first discovered by a meteorologist called Edward Lorenz in 1960. Lorenz was studying a particular set of 12 differential equations and their numerical solutions for a given set of initial conditions, which he thought, might be able to predict certain weather conditions. That in its self is not peculiar in the slightest. Mathematicians had been investigating differential equations of all shapes and sizes numerically for a very long time. The only difference between how Lorenz was conducting his investigations and how most people up until that time investigated the equations, was that Lorenz had a small "personal computer" at his disposal, that could turn days and days of horrible arithmetic and algebraic drudgery for even just a small amount of numerical data, into a few hours. His ability to process these equations numerically, far exceeded that of "classical" mathematicians, which is the very reason why Chaos Theory was not discovered earlier than it was. Chaos theory required a computer to be seen.

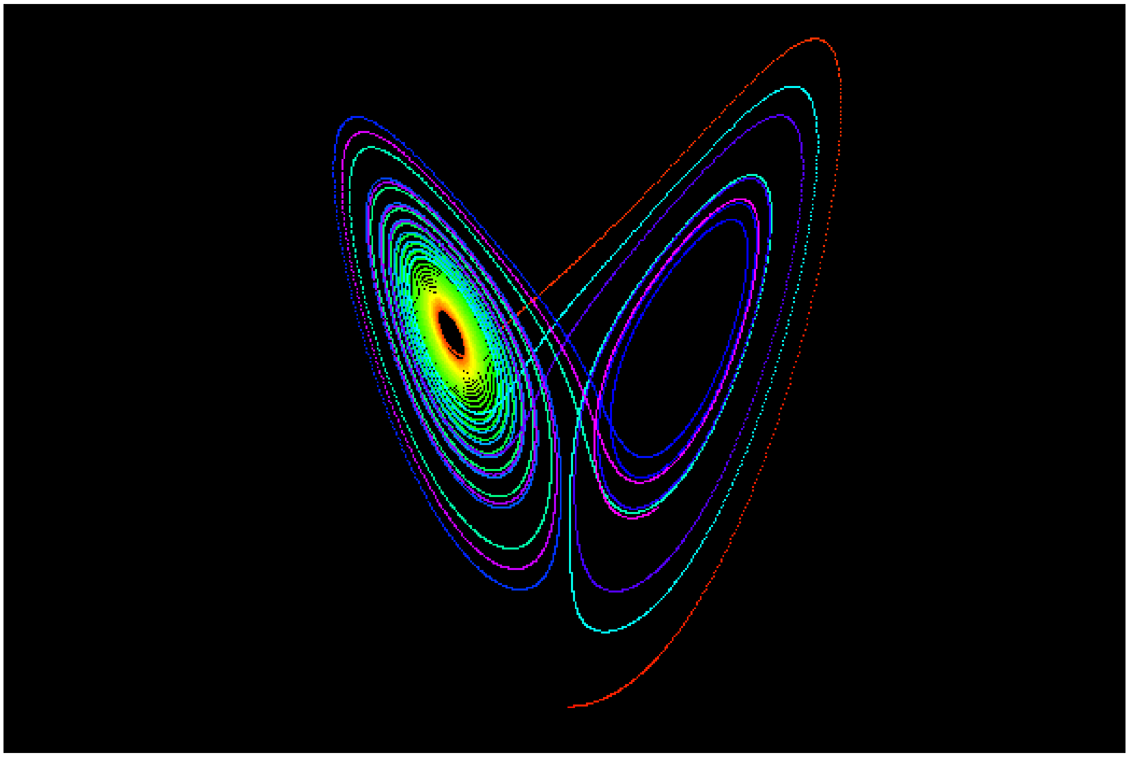
So what was so strange about Lorenz's investigations? Lorenz noticed that if he solved the equations with a particular set of initial conditions, and then went back and did all the calculations again with a slightly varied set of initial conditions, the end results were extraordinarily different. What he had discovered was a sort of "Sensitivity" of these equations to their initial conditions. How did he actually see this variation? Lorenz wished to see a particular sequence of numbers again. He had previously generated a print out of this sequence of numbers that came from solving these differential equations, but wished to generate them again. To save himself some time, instead of starting from scratch and generating all the numbers again, he thought he would use his old sequence and read off a number from the middle of the sequence and let it run from there. This way he would only have to wait half the time to generate the same sequence of numbers he already had. However, his computer stored numbers to six decimal places, and Lorenz, thinking it would not matter, only entered three decimal places. When he came back after a couple of hours, he noticed that the sequence was completely different to that he had previously obtained. What he noticed was a sensitivity to initial conditions, and what is now lovingly termed "The butterfly effect".

The butterfly effect is usually explained by describing a butterfly that flaps its wings, and causes a cyclone on the other side of the world. The essence of the analogy is that tiny variations cause huge effects. In this case it was the small pressure variations from the butterfly's wings that ended up generating huge weather effects miles away. The actual Mathematical description of the butterfly effect, we have just explained. All it is, is that a slight variation of initial conditions inputted into the numerical solution of certain equations, can lead to drastically varied results. This was the beginning of chaos theory, and the embryo of fractals.

Looking at the Order in Chaos

From this starting point Lorenz then started to look at a simpler situation involving only 3 differential equations. What he did was take the equations of Convection and strip them down until they were very simply related, obtaining only 3 equations. It would later be shown that these equations described a water wheel.

This new set of equations also showed this sensitivity to initial conditions, however this time when he graphed the solutions for this system (in what's known as Poincare space, or phase space), he noticed that the output was always situated on two curves.



In what seemed to be an amazing contradiction in terms, there appeared to be some sort of order within the chaotic structure of the numbers generated from these equations.

Similar systems that are sensitive to variations in initial conditions exist in many different areas. Most notably is the classic example in biology and population dynamics. The standard equation of such areas is most simply stated as

$$P_{n+1} = aP_n(1 - P_n)$$

Where

P_n is the previous years population, and is given on a scale from 0 to 1, thus

P_{n+1} is the next years population, and a is a constant called the growth rate.

Such an equation is called a recursive equation, meaning that it will tell you the next lot of values based on the previous values. Most differential equations can be reduced to this type of equation for numerical evaluation.

To see more clearly how this type of equation works lets plug in a few experimental values. Lets say that at year 0 we have a population of .5, and we put a=1 so we can write,

$$P_0 = \frac{1}{2}$$

These are our initial conditions. Then to determine the next years population I will have to use,

$$P_1 = P_0(1 - P_0)$$

which is just from our main formula . Thus substituting we get

$$P_1 = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$$

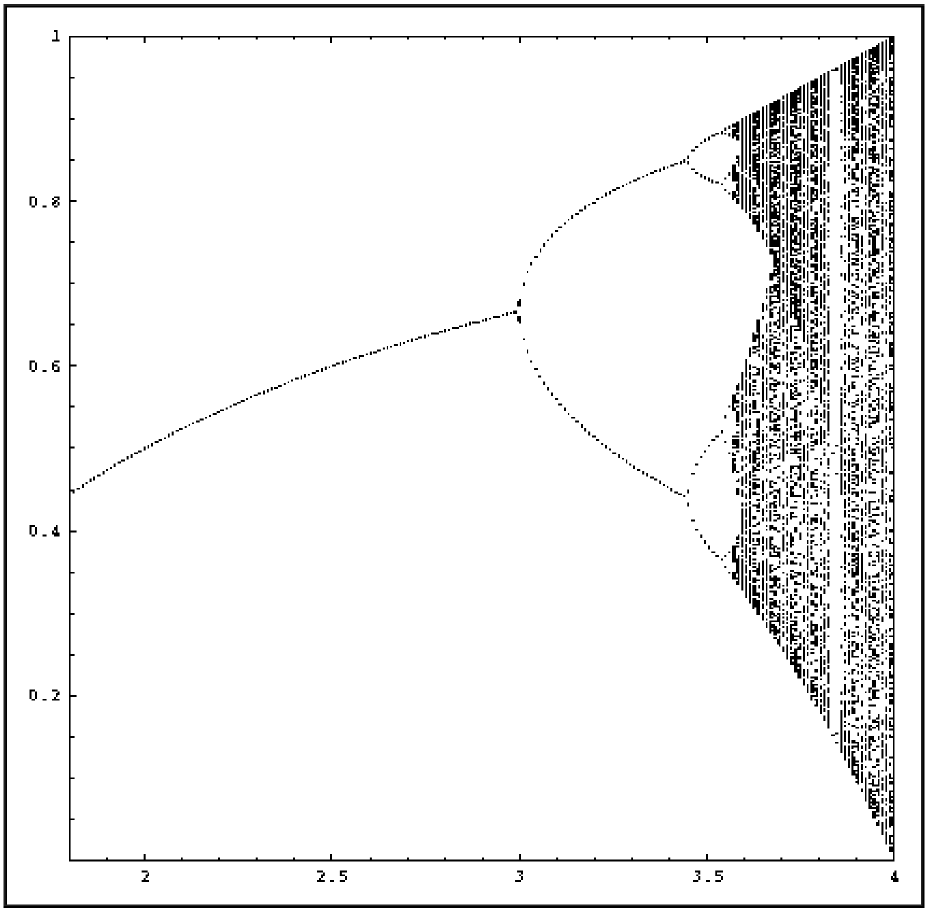
and thus

$$P_2 = \frac{1}{4}(1 - \frac{1}{4}) = \frac{3}{16}$$

and so on and so on. This process by where we determine the next data point from the previous data point is called iteration. The process of iteration is at the very heart of chaos and fractals, it is how the differential equations are solved, and it is how fractals are generated. In the above example we looked at some iterations for a particular growth rate of a=1, but what happens if we change this growth rate?

What happens to the sequence of numbers that are generated by iteration for different rates of growth?

A biologist named Robert May, investigated such questions for this particular equation. He found that if the growth rate was a low number then the iterative process would converge to ONE number. This means that after we perform enough iterative steps, we just keep on getting the same number again and again. Then something very strange happened. He noted that after say about a=3, the equation would not converge to just one number, but converged to two numbers! How can this happen? He noticed that after some iteration the sequence converged to one particular number for a particular year, then the next iteration it was a different number, then the following iteration it returned to the number for the year before. So it was oscillating between two different numbers. More simply stated; if you put one of these numbers into the equation you got the other number. As he further increased the growth rate parameter, the sequence split again (bifurcation.....doubled) and oscillated between four different numbers. The further he increased the growth rate the faster the bifurcation occurred, until it reached a point where it became totally chaotic and the system became unpredictable.



In the above diagram (known as an orbit diagram) we see the values where the iteration converges to a single value. Then further on we see it split (the first bifurcation). A little further on we see it double again, and then again. Then we see the horrible dark mess that is chaos.

If we look a little further on, we see a white strip. In actuality (although you might not be able to make it out) this white strip is where the whole process starts again. We again have single values which intern double, then double again, then once more return to the chaos regime. You will notice there are a few white strips in the diagram. Therefore we are staring at a diagram that actually contains repetitions of ITSELF WITHIN ITSELF. It is this self-similarity that became a very important feature of chaos theory. This should be reminding you of something very familiar by now. In fact it is the one-dimensional version of a very famous object. But more on that later.

There is so much more interesting stuff we could go into about chaos theory, just given the very basics we have covered here. For instance there is the Feigenbaum number, and its relation to the scales of the period doubling areas in the orbit diagram. It is a number that is constant and exists in all iteration schemes such as this. More mysteriously it is an irrational number, and one particular Feigenbaum number pops up in certain phase transitions in physics, and is very nearly equal to

$$\pi + \arctan(e^{-\pi})$$

Astronomers have also witnessed radio signals from space that have peaked from the background noise and are almost exactly equal to the Feigenbaum constant up to approximately 10 decimal places. This is merely the very tip of the iceberg as far as chaos theory goes. There are many more strange properties contained with in its fold, and one of these just happens to be the world of fractals.

The Birth of Fractals

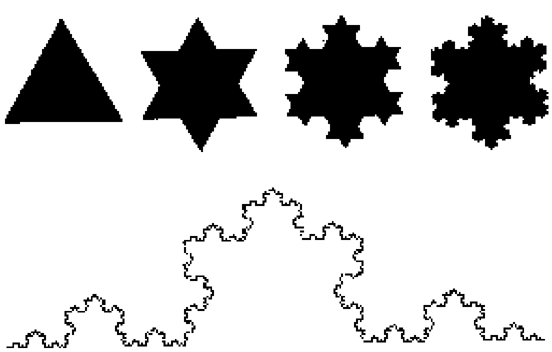
Up to this point we have discussed the main underlying nature of chaos theory, a theory that was born from a natural mistake of omitting a few digits in a numerical computation. Never again were scientists to be so haphazard. However at this stage chaos theory really wasn't "welcomed with open arms" into the scientific community. It was more of a hobby that biologists like to work on now and again and the "lets see what we can come up with" attitude dominated the area. The reason for such semi-rejection from the scientific community was largely based upon the fact that chaos theory at the time, didn't really have a real world problem that it could help solve. It was more of an odd curiosity. And like most of the greatest theories in science, it was at first overlooked.

Poincare had done some previous work in dynamics that touched on chaos theory, and it is to him that we owe the Poincare phase space, where so many chaotic systems are graphed. However he never really got the chance to peer at the numerical intricacies of the subject, simply because he didn't have the computational power to do so at the time.

Cantor on the other hand, I feel to be one of the first persons to glimpse the edge of chaos theory. Cantor was a brilliant mathematician who dedicated his life to studying numbers and the theory of sets. In particular he was the first person to come up with the idea of infinite numbers, and proved the fact that it was indeed possible to have one infinity larger than the other. He too was rejected by the scientific community, however his theories are no considered not only the cornerstone of certain areas of mathematics, but form the building blocks of mathematics itself! And given the nature of his work and the fact that it is probably the most detailed definition of infinity the human race has known it is no surprise that we should find him connected in some way to chaos theory. But how exactly?

The connection comes about from considering self-similarity. Benoit Mandelbrot, an employee of IBM at the time, was considering this very phenomenon of self-similarity, which he had stumbled upon by analyzing cotton prices. He noticed that although the price fluctuations of cotton over the years, never really fitted on a normal distribution, their scale was almost identical. What he saw was that while the daily and monthly pricing of cotton were random, the curves of daily price changes, and monthly price changes were identical. This self similarity with scale lead him to consider other such problems, such as the self-similarity of a coast line. If we look at a coast line on a map, we see bays etc, but we don't see the small bays that are within the bays. We also don't see the small structures that are within those bays and so on.

This idea was more mathematically demonstrated by Helge Von Koch. He constructed a curve now known as the Koch curve. To construct a Koch curve, you start with an equilateral triangle and add another equilateral triangle to the middle third of each side of the original triangle. Then you repeat this ad-infinitum. What you end up with is a structure that if you zoom in on any area of the curve, it looks exactly like the overall curve.



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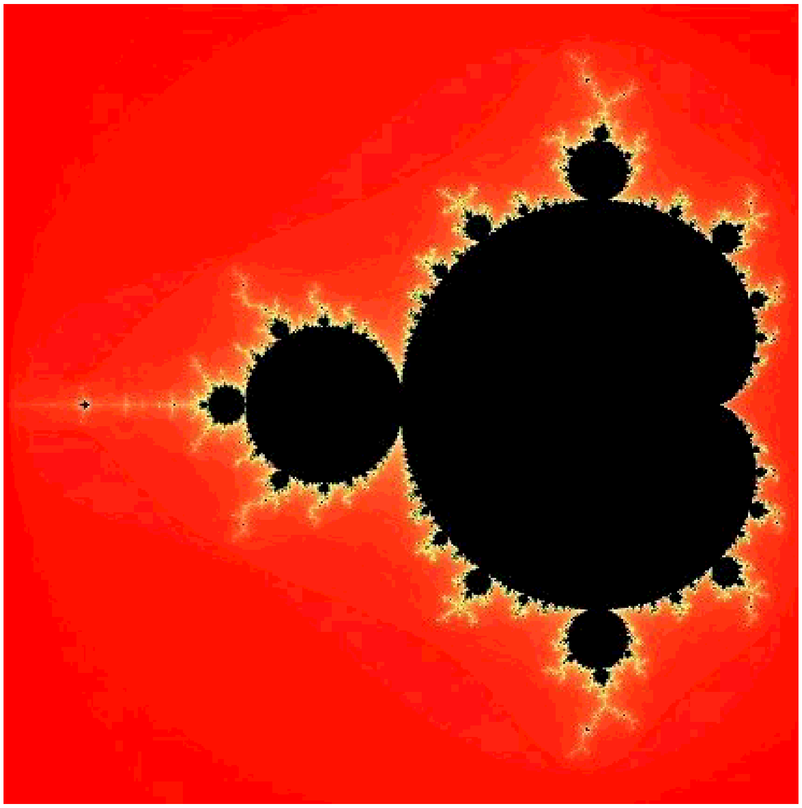

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