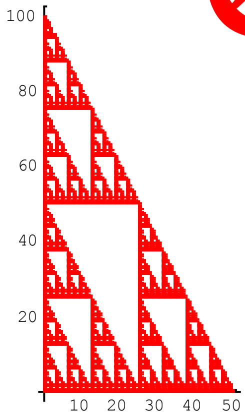
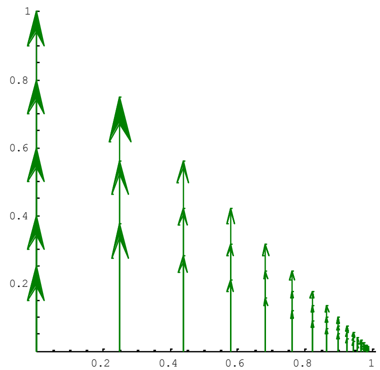
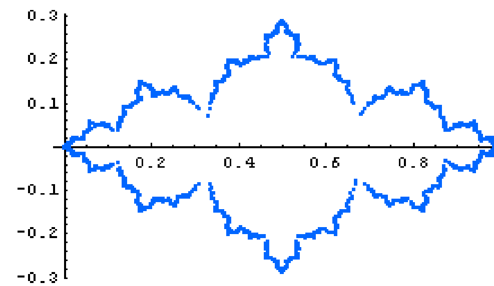
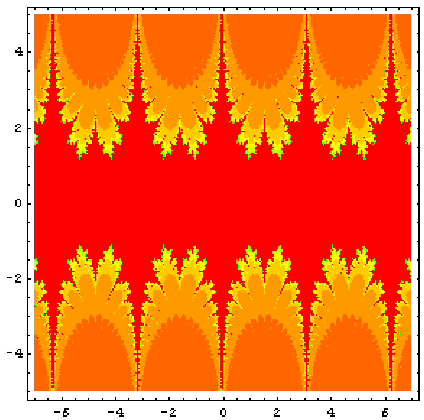
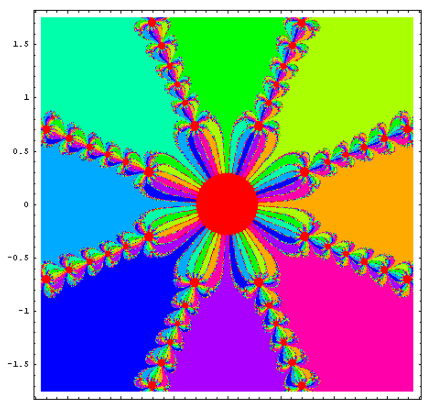
CHAOS / FRACTAL

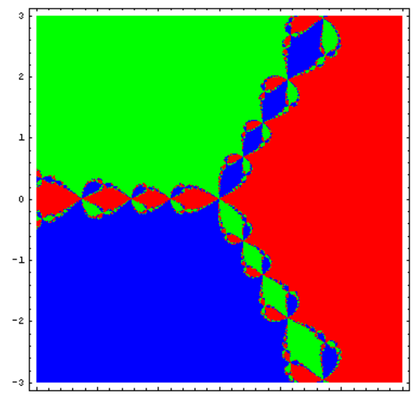


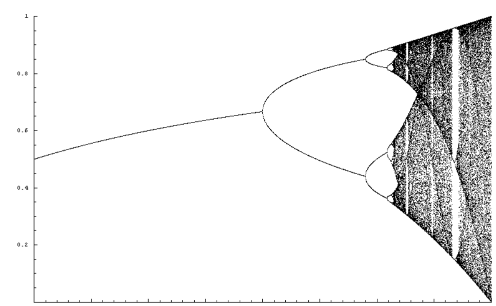


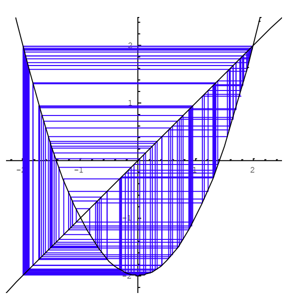


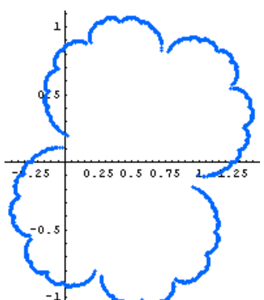


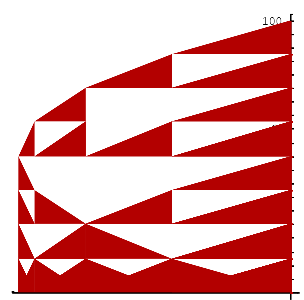


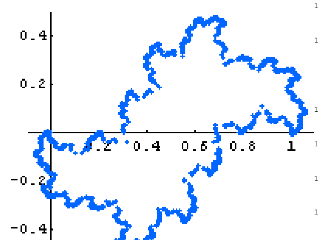


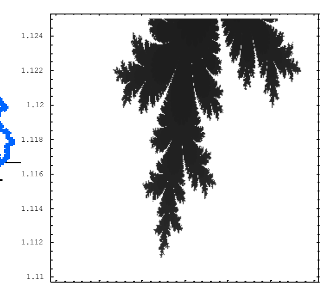


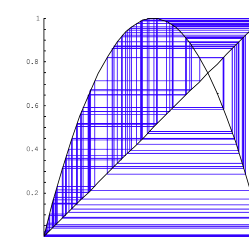












Presentation in Dynamical System.

- By Fred Khoury

Introduction

This paper is divided into three sections:

1. The behavior of the logistic and quadratic functions in real plane and real parameters. Then the behavior of the logistic function in complex orbit with real and complex parameters.

2. The fractal curve of COSINE function in complex number.

3. Generating trees, a Sierpinski triangle, and a Fern branch through Iterated Function System (I.F.S).

1- Part A - Logistic Function: $g_\lambda(x) = \lambda x\,(1 - x)$

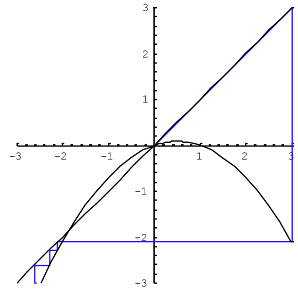
The fixed point(s) is/are when the function g_λ (x) intersects with the bisector equation F(x) = x, where the parameter $\lambda \in$ ☼.

That imply to $g_\lambda(x) = F(x) \Rightarrow \lambda x\,(1 - x) = x$

$\Rightarrow$ $-x\,(\lambda x + 1 - \lambda) = 0$

$\Rightarrow$ The fixed points are: x = 0, and $x = \dfrac{\lambda - 1}{\lambda}$

Example: $\lambda = .3$



The stability of a fixed point is dependent on the nearby behavior that either converges to or diverges from the fixed points.

$$\frac{d}{dx}\,[g_\lambda(x)] = \lambda(1 - 2x)$$

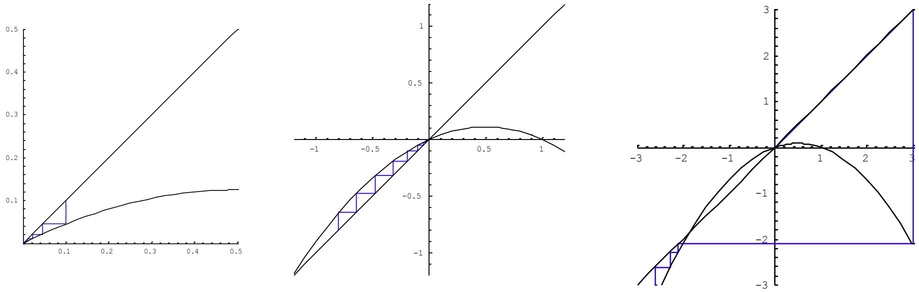
- $\dfrac{d}{dx}\,[g_\lambda(0)] = \lambda$

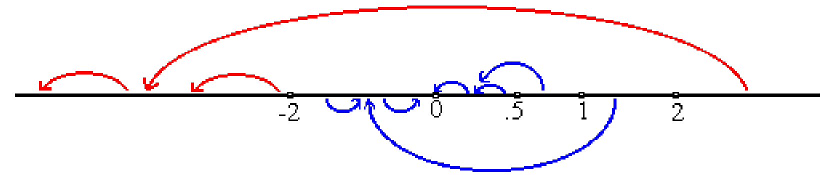
if $|\lambda| > 1 \Rightarrow \left|\dfrac{d}{dx}\left(g_\lambda(0)\right)\right| > 1$; then the fixed point x = 0 is unstable

if $|\lambda| < 1 \Rightarrow \left|\dfrac{d}{dx}\left(g_\lambda(0)\right)\right| < 1$; then the fixed point x = 0 is stable

if $\lambda = 1 \Rightarrow \left|\dfrac{d}{dx}\left(g_\lambda(0)\right)\right| = 1$, the fixed point at x = 0 is undetermined

- The same analysis goes to $x = \dfrac{\lambda - 1}{\lambda}$

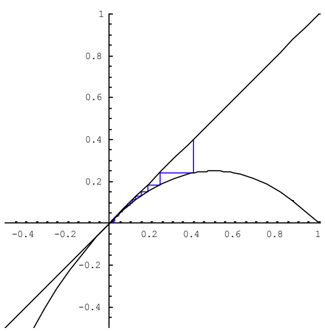


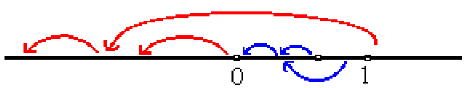


Graphical analysis $\lambda = .5$ and the starting point = .1, -.8, and .3 respectively

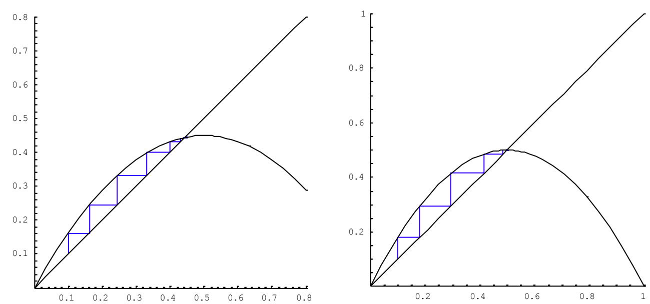
Phase portrait of $g_\lambda(x)$ when $0 < \lambda < 1$

If $\boldsymbol{\lambda = 1}$, there is only one fixed point at $x_{1,2} = 0$, where $g_\lambda(x)$ is neither attracting nor repelling.

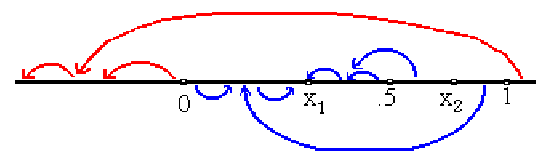




Phase portrait of $g_\lambda(x)$ when $\lambda = 1$.



Graphical analysis of $g_\lambda(x)$ when $1 < \lambda < 2$.

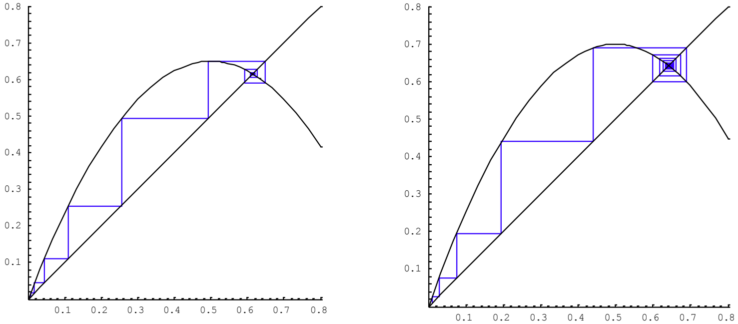


Phase Portrait of $g_\lambda(x)$ when $1 < \lambda < 2$.

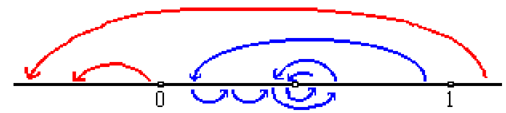
But when $\boldsymbol{\lambda > 1}$, the logistic function $g_\lambda(x)$ intersects the bisector function at $x = 0$ and $x = \dfrac{\lambda - 1}{\lambda}$

$$\frac{d}{dx}\,[g_\lambda(x)] = \lambda(1 - 2x) \;\Rightarrow\; \left|\frac{d}{dx}\big(g_\lambda(0)\big)\right| = \lambda > 1.$$

Therefore the logistic function $g_\lambda(x)$ is unstable at $x = 0$ for all $\lambda > 1$. $g_\lambda(x)$ is repelling from now as $\lambda > 1$.



Graphical analysis of $g_\lambda(x)$ when $2 < \lambda < 3$.



Phase Portrait of $g_\lambda(x)$ when $2 < \lambda < 3$.

However, when $2 < \lambda < 3$, the iteration converges to $\frac{\lambda - 1}{\lambda}$ but by going in loop until it attracts it.

$$g_\lambda^2(x) = \lambda^2 x (1 - x)(1 - \lambda x + \lambda x^2) = 0$$

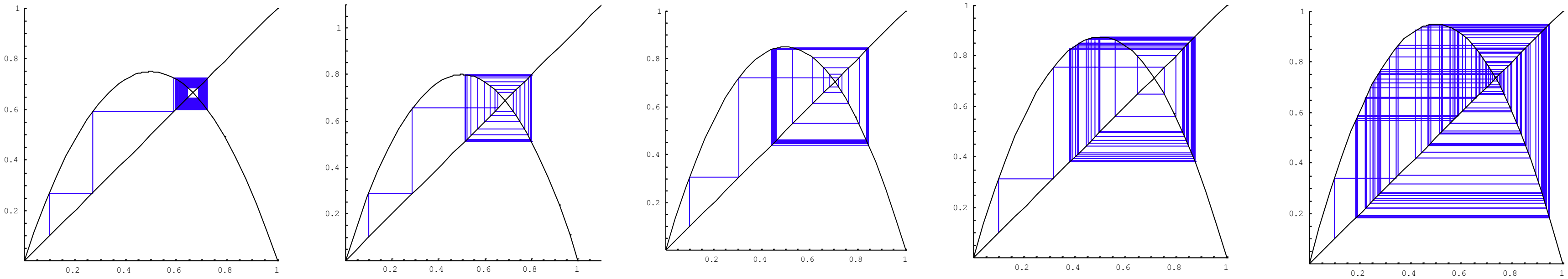
So, for the roots to be real numbers: $\sqrt{\lambda^2 - 2\lambda - 3} > 0 \Rightarrow \lambda < -1$ and $\lambda > 3$.

$$\left|\frac{d}{dx}\left(g_3(0)\right)\right| = 3 > 1 \quad \Rightarrow \text{ the system is repelling at 0.}$$

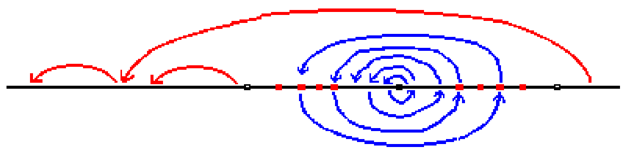
But for **$\lambda = 3$**, the logistic becomes $g_3(x) = 3x(1 - x)$ and the fixed points are 0 and $\frac{2}{3}$.

As λ passes through 3, a single attracting fixed points becomes an attracting 2-cycle.

Therefore, the derivative of g_3 at the fixed point $\frac{2}{3}$ is equal to -1, indicating a period-doubling bifurcation at $\lambda = 3$.



$\lambda =$ 3. 3.2 (10000 iterations) 3.4 3.5 3.8

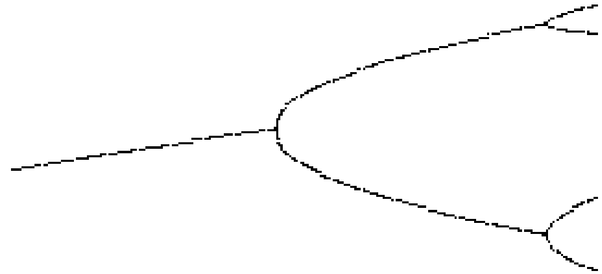


Phase Portrait of $g_\lambda(x)$ when $3.45 < \lambda < 4$.

For the third iteration: $g_\lambda^3(x) = \lambda^3\, x\, (1-x)(1 - \lambda x + \lambda x^2))\, (1 - \lambda^2 x + (\lambda^2 + \lambda^3)x^2 - \lambda^3 x^4 - 2\,\lambda^3 x^3)$.

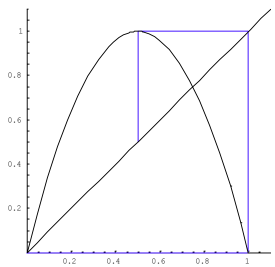
When $\lambda > 3.44940...$, the function with power degree of 8.

Therefore, the dynamical behavior is a period-doubling bifurcation with period 4-orbit.

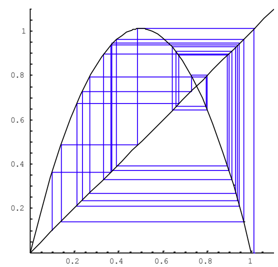


pitchfork diagram.

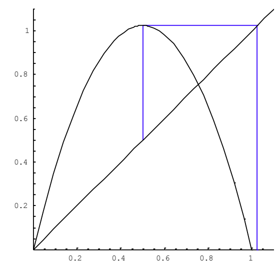
As we see, that every branch is splitting/divided into two branches or periodic orbits or cycles. The terminology of that, the fraction play a role by dividing every cycle into two separate cycles (1 to 2 to 4 to 8 … to ½ 2^n) as shown above.



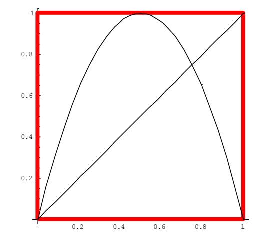
λ = 4.



4.1



4.1

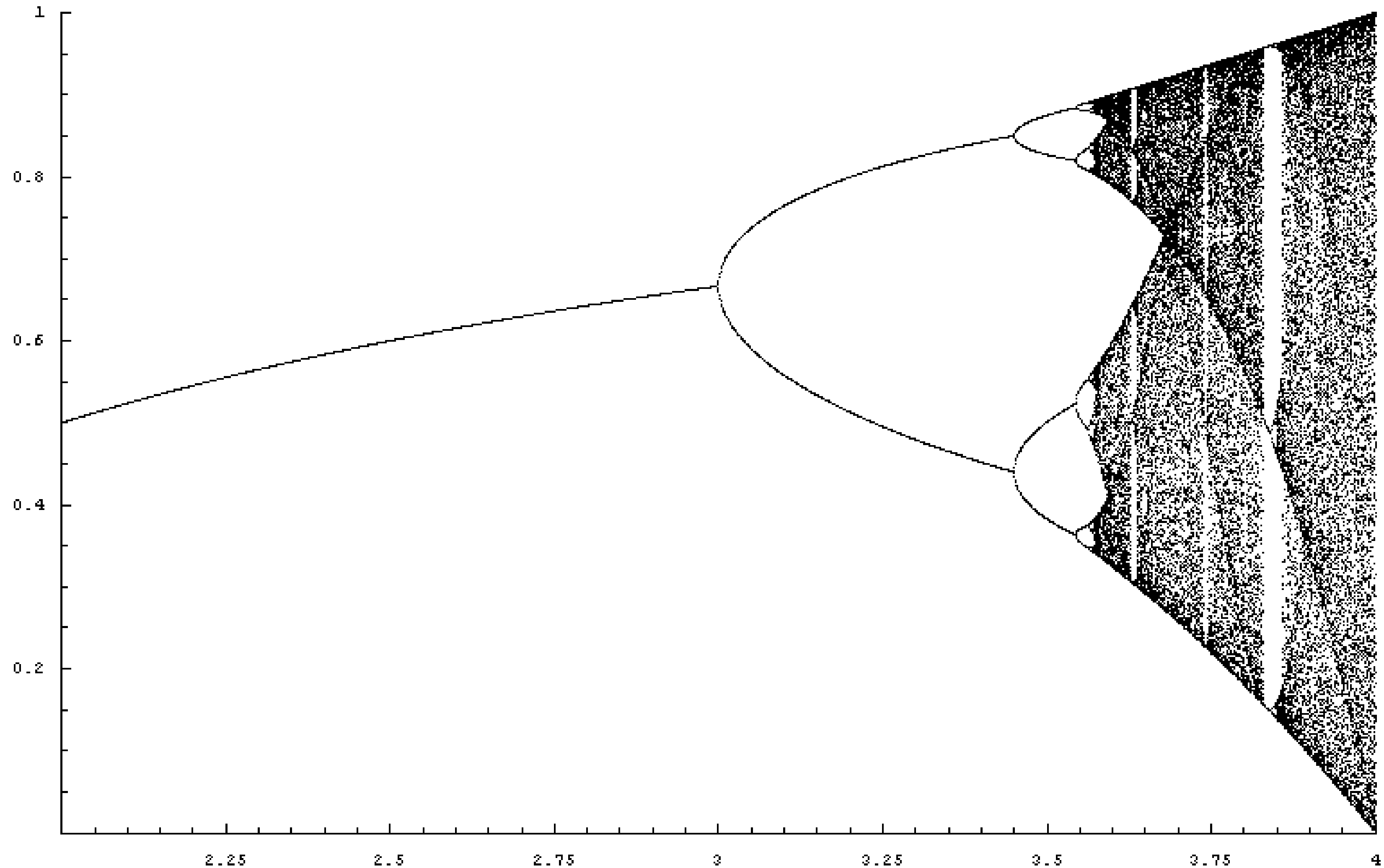


Boundary of the dynamical behavior of logistic function.

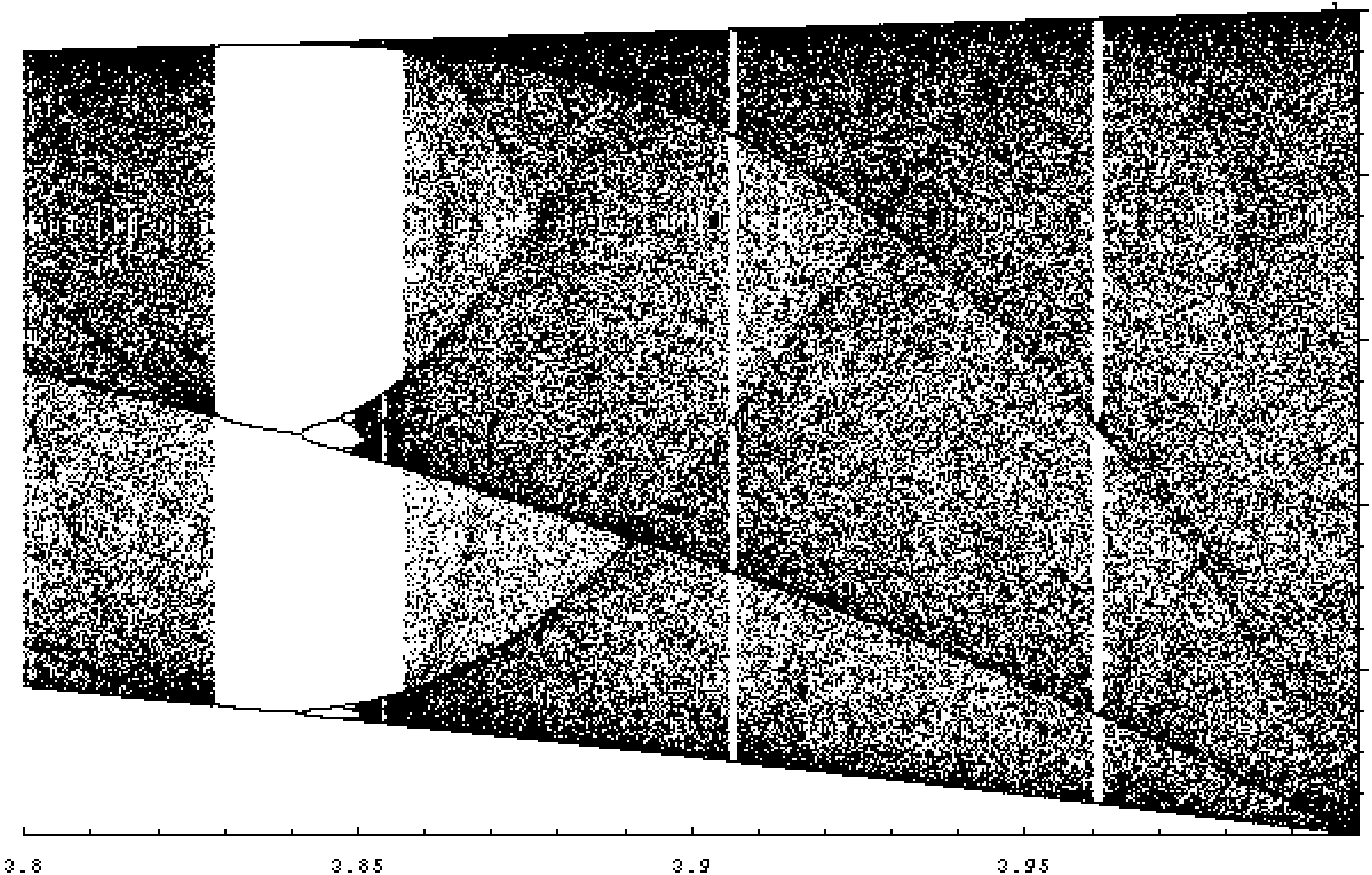
“The word „fractal“ from the Latin root fractus, suggesting fragment, broken and discontinuous.”

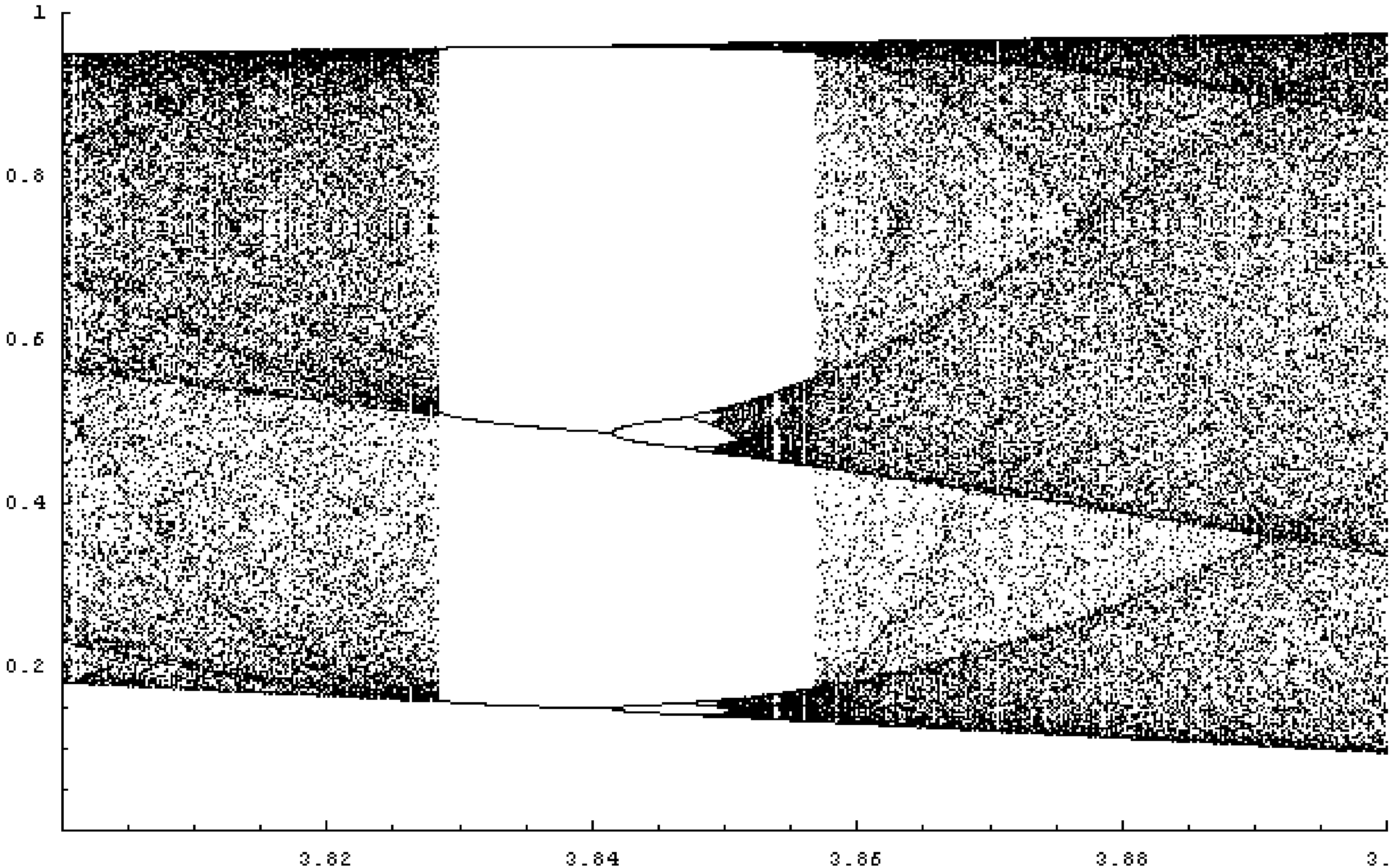
This word was described by Benoit Mandelbrot in 1975 is:

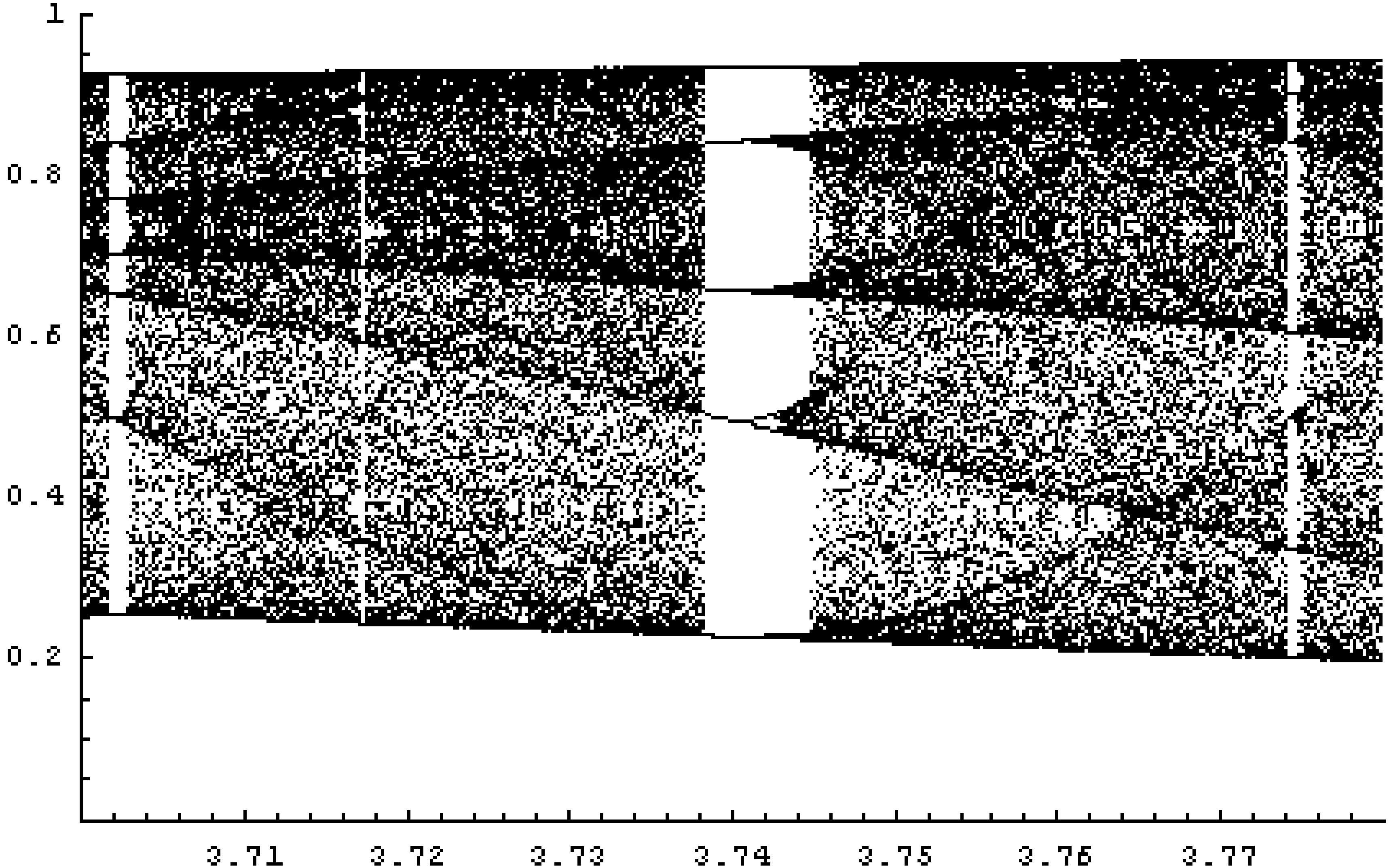
“To describe shapes which are detailed at all scales. “



Feigenbaum diagram.


3.8
3.85
3.9
3.95


1
0.8
0.6
0.4
0.2
3.82
3.84
3.86
3.88
3.9



1
0.8
0.6
0.4
0.2
3.71
3.72
3.73
3.74
3.75
3.76
3.77

Part B - The quadratic function: $f_\mu(x) = x^2 + \mu$

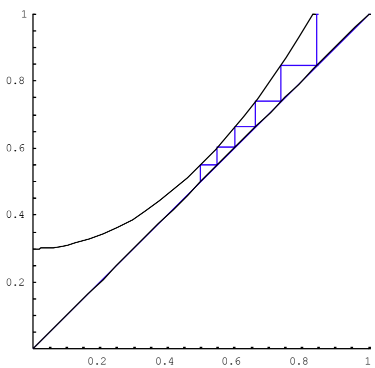
The fixed points for the quadratic function $f_\mu(x)$ are given by:

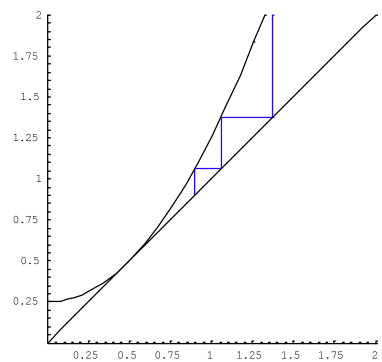
$f_\mu(x) = F(x) \Rightarrow x^2 + \mu = x \Rightarrow x^2 - x + \mu = 0$, where $\mu \in$ ☼.

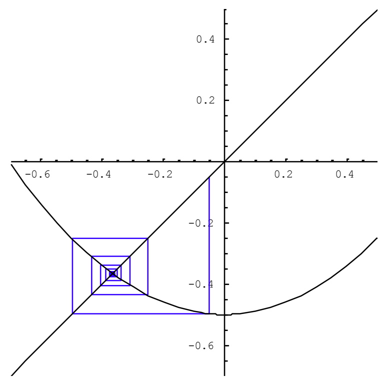
The roots of the quadratic equation are: $x_{1,2} = \frac{1 \pm \sqrt{1-4\mu}}{2}$

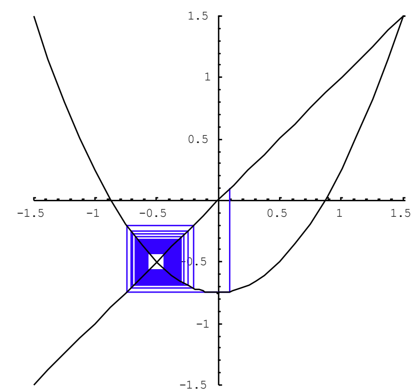
We can rewrite these fixed points in the form of: $P+ = \frac{1+\sqrt{1-4\mu}}{2}$, and $P- = \frac{1-\sqrt{1-4\mu}}{2}$

If $(1 - 4\mu) < 0 \Rightarrow \mu > ¼$, then the fixed points are imaginary and since the fixed points are real.









The second iteration of $f_\mu(x)$ is: $f_\mu^2(x) = (x^2 + \mu)^2 + \mu$

$\Rightarrow$ Therefore $f_\mu^2(x)$ equation has 4 roots.

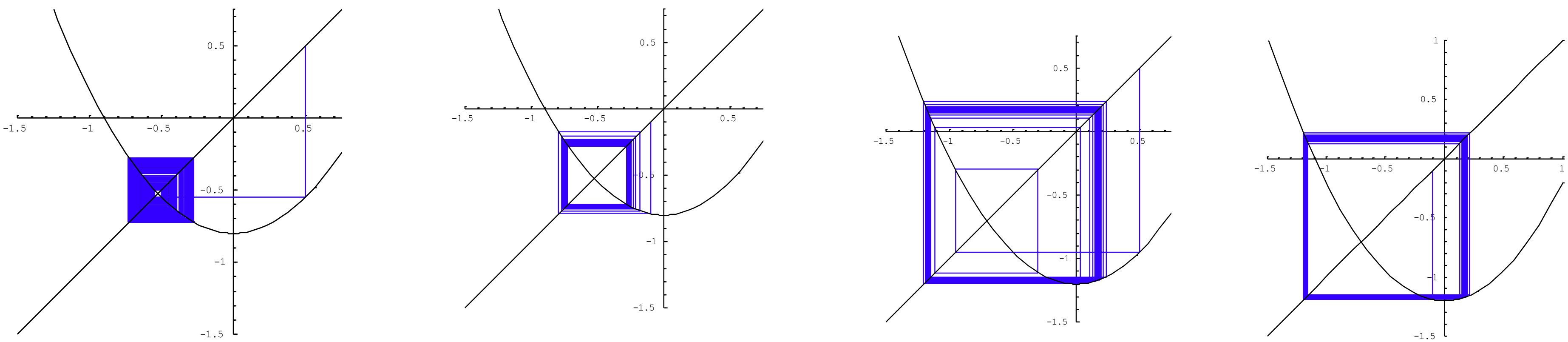
$\Rightarrow$ P_-, P_+, $P_{1-} = ½ (-1 - \sqrt{-3-4\mu})$, and $P_{1+} = ½ (-1 + \sqrt{-3-4\mu})$; which are periodic points of period 2 for $f_\mu(x)$.

At P_{1+} and P_{1-} $\Rightarrow$ $\left|\frac{d}{dx}\left[f_\mu^2(P_{1\pm})\right]\right| = \left|4\,(1+\mu)\right| < 1$ is stable,

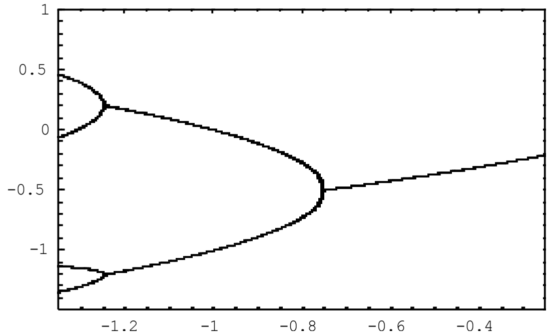
$$-1 < 4\,(1+\mu) < 1 \;\Rightarrow\; -\frac{1}{4} < (1+\mu) < \frac{1}{4} \;\Rightarrow\; -\frac{5}{4} < \mu < -\frac{3}{4}$$

Therefore for $-\frac{5}{4} < \mu < -\frac{3}{4}$ is stable and attracting with 2-cycle (periodic of 2-period orbits),

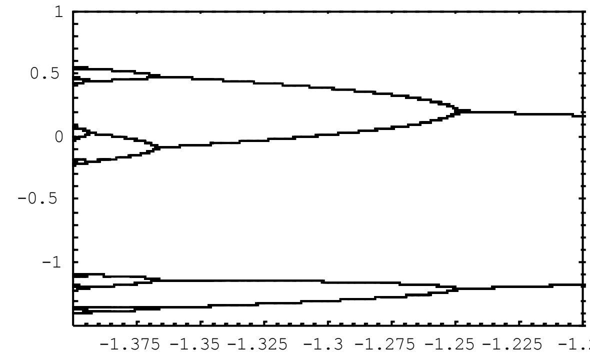
or it is called <u>period-doubling bifurcation</u>; otherwise is unstable.



μ = -.8 (Starting point 0.5) -.8 (Starting point –0.1) -1.0 -1.2

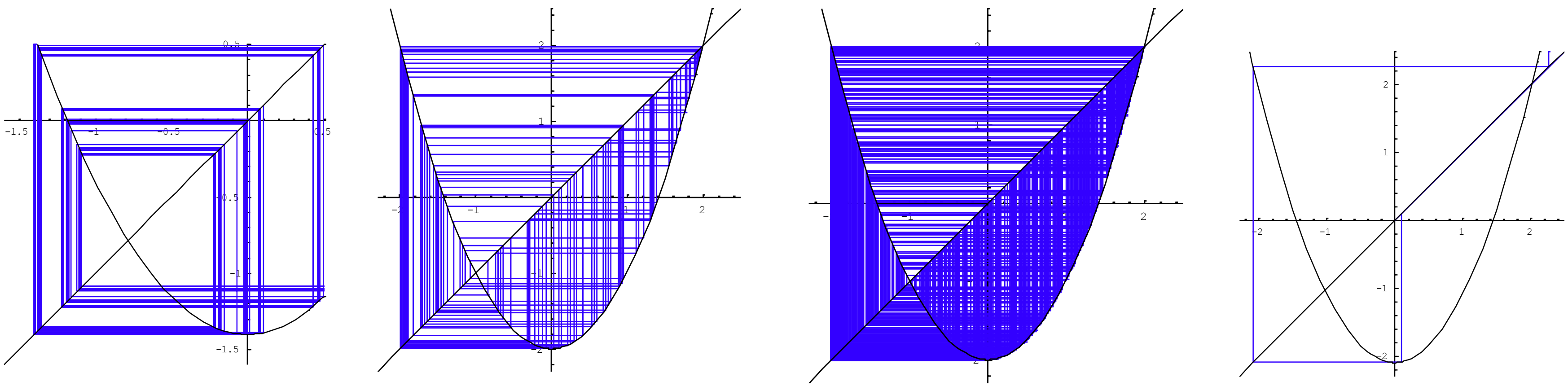


(a) 4-cycles

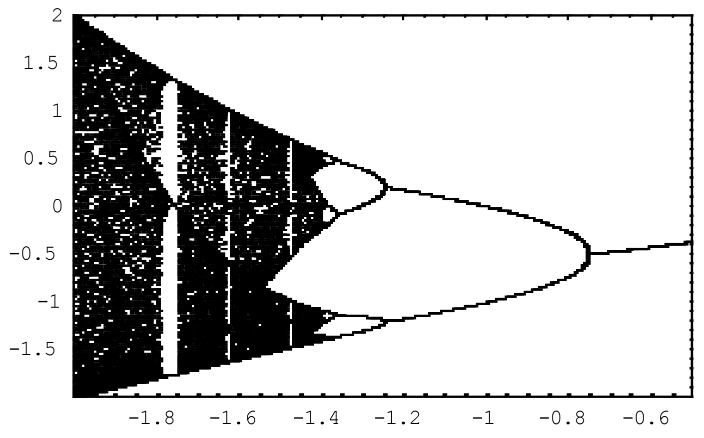


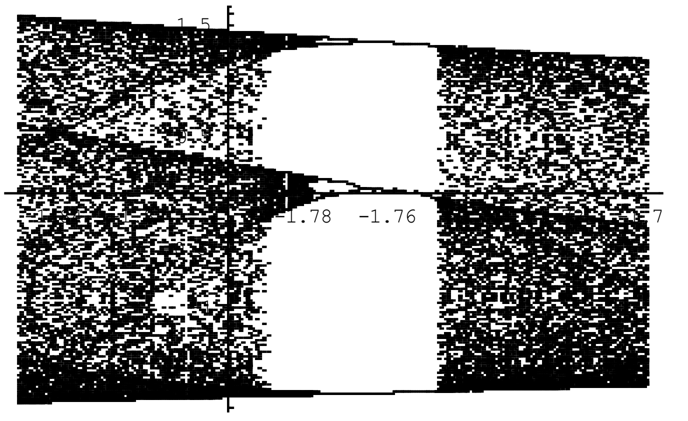
(b) more cycles

Pitchfork bifurcation diagram of $f_\mu(x)$



Graphical analysis of $f_\mu(x)$ when $\mu =$ -1.4 , -2., -2.(500 iterations), -2.1.



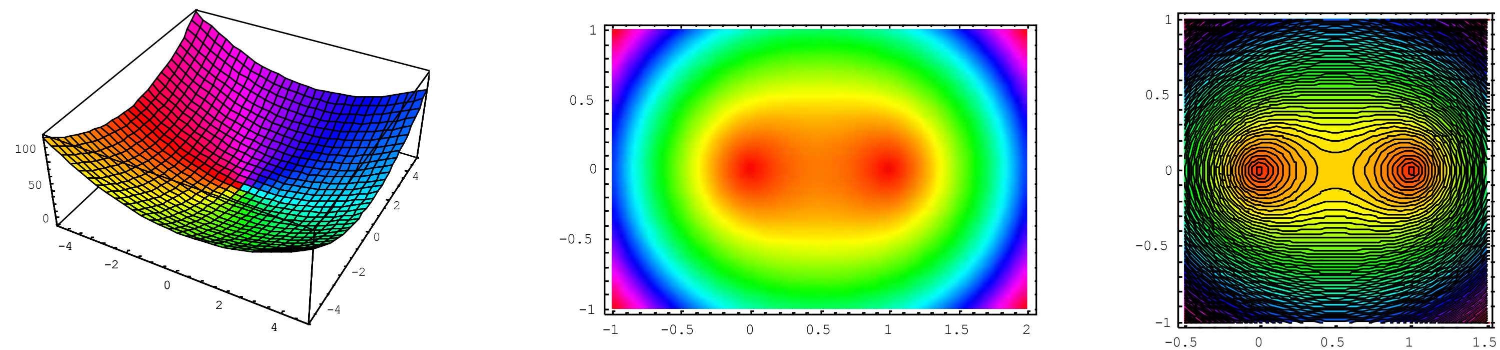


Feigenbaum diagram.

When μ is closed to ≈-1.755, the dynamical behavior changes the period attractor to period 3, 5, 7, and so on (all exist), replacing the period of 2, 4, 8, 16, …. That why, the period-3 attractor is causing that window to open in the bifurcation diagram.

Part- C. The logistic function $f(z) = \lambda z (1 - z)$

Testing the behavior of a function and determining the birth of a fractal curve, is time-consuming. A quicker method for calculating points on the fractal curve is to use the inverse of the function. Any initial point chosen either from inside or outside of the curve will then converge to the points on the fractal curve.



Plot for $f(z) = \lambda z (1 - z)$.

Therefore, the fixed points of f(z) are: $z = 0$, and $z = \dfrac{\lambda - 1}{\lambda}$

The stability of this system depends on the value of λ. When λ is real, this behavior had been determined in the logistic function of the real function $g_{\lambda}(x)$ earlier.

Let determine the inverse function of f(z) such that: $f(z) = \lambda z(1 - z) = \lambda z - \lambda z^2 \Rightarrow \lambda z^2 - \lambda z + f = 0$

The roots of the second degree function of z are given:

$$z_{1,2} = \frac{\lambda \pm \sqrt{\lambda^2 - 4\lambda f}}{2\lambda} = \frac{1 \pm \sqrt{1 - 4f/\lambda}}{2} = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\frac{f}{\lambda}}\right)$$

Therefore the inverse function can be obtained from the complex roots, thus the are two transformation functions is in the form of:

$$C(z) \text{ or } z = \frac{1}{2}\left(1 - \sqrt{1 - 4\frac{z}{\lambda}}\right) \quad \text{and} \quad z = \frac{1}{2}\left(1 + \sqrt{1 - 4\frac{z}{\lambda}}\right)$$

For $\underline{\boldsymbol{\lambda = 3}}$ ($\in \Re$); the fixed point of f(z) are 0 and $\frac{2}{3}$.

$\left|\frac{d}{dz}(f_\lambda(0))\right| = |\lambda| = 3 > 1$; then the system is unstable repelling at the origin.

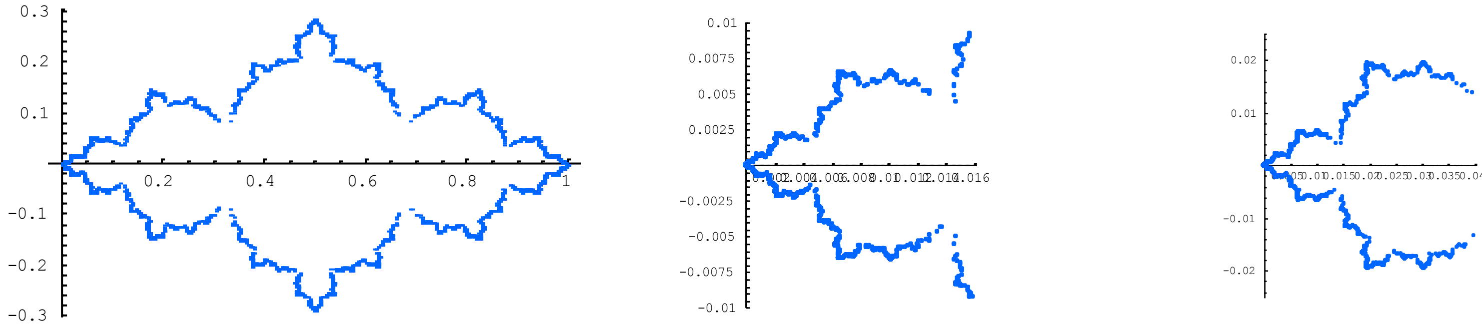
$\left|\frac{d}{dz}\left(f_\lambda\left(\frac{\lambda - 1}{\lambda}\right)\right)\right| = |2 - \lambda| = 1$; then the system is undetermined at $\frac{\lambda - 1}{\lambda}$.

Recall the logistic function, the bifurcation in the orbit behavior takes place near $\lambda = 3$.
An attracting fixed point becomes a repelling fixed point and becomes an attracting 2-cycle (period- doubling bifurcation at $\lambda = 3$).

Let apply the fixed points to the complex function C(z):

$$C(0) = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{3}0}\right) = \frac{1}{2}(1 \pm 1) = \begin{cases} 0 \\ 1 \end{cases}$$

$$C\left(\frac{2}{3}\right) = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4}{3}\frac{2}{3}}\right) = \frac{1}{2}\left(1 \pm \frac{1}{3}\right) = \frac{1}{3} \text{ and } \frac{2}{3}$$

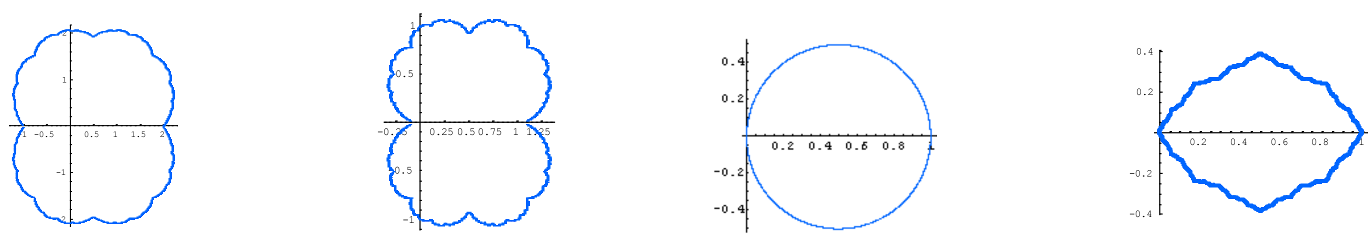


The inverse function when $\lambda = 3$ - zoom-in. „cusp-like „

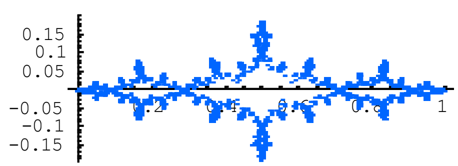
$$C'(z) = \pm\frac{1}{3}\left(1-\frac{4}{3}z\right)^{-.5}$$

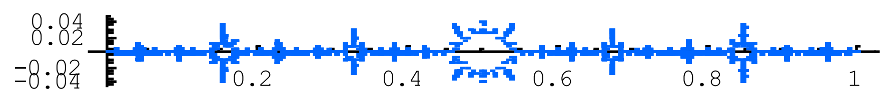
$C'(\frac{2}{3}) = \pm\frac{1}{3}\left(1-\frac{4}{3}\frac{2}{3}\right)^{-.5} = \pm 1$, the system attracting near the fixed point $\frac{2}{3}$.

$C'(\frac{1}{3}) = \pm\frac{1}{3}\left(1-\frac{4}{3}\frac{1}{3}\right)^{-.5} = \pm\frac{1}{3}\left(-\frac{1}{3}\right)^{-.5} \in$, the system is repelling periodic points near the fixed point $\frac{1}{3}$

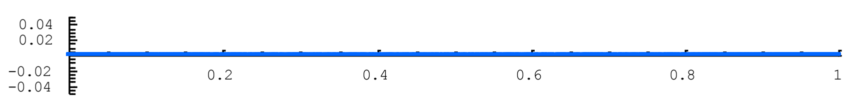


$\lambda =$ .5 1. 2. 2.5





$\lambda =$ 3.5 3.89

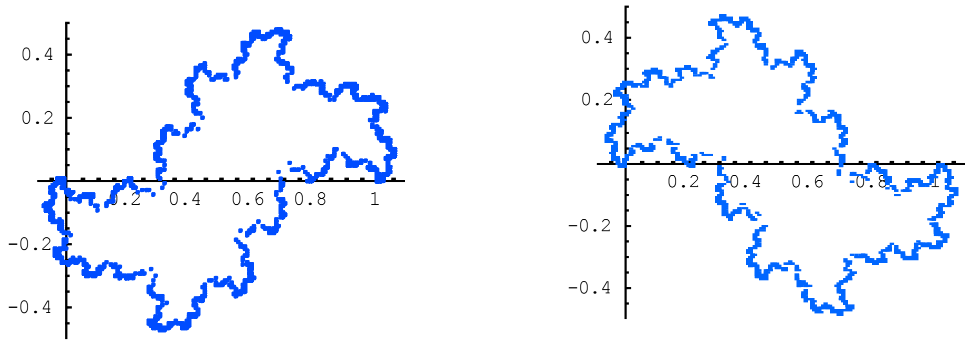


$\lambda = 4.$

For $\lambda \in \mathbb{C}$;

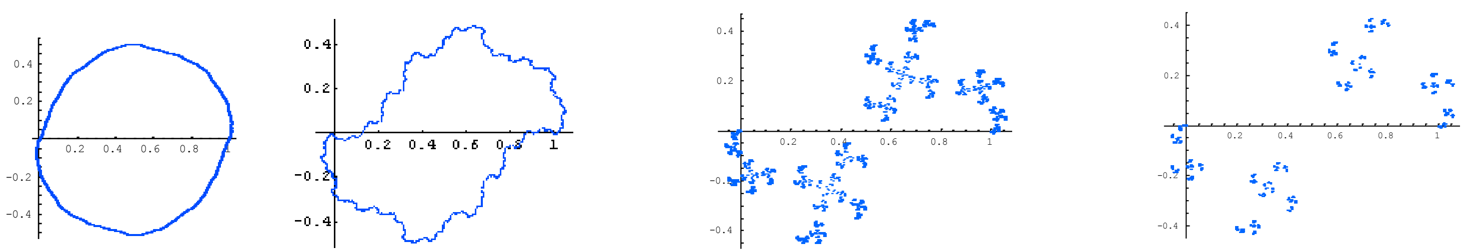
Let $\lambda = 2 + ai$; where $a \in \mathbb{R}$.
Let choose first the value for: $a = 1 \Rightarrow \lambda = 2 + i$.
The fixed points of the system for this parameter λ are: 0 and $\frac{1}{5}(3 + i)$.

When the inverse for this transformation is iterated, the interior points converge on the boundary of the set, and similarly the exterior points converge to the same set, unless otherwise already on the set. This set or boundary seems that it attracts the orbits of all points of the z plane.



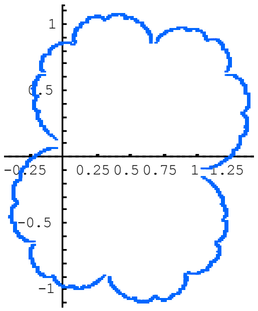
The set of Julia after 10000-point: for $\lambda = 2 + i$ and $2 - i$

These iterations have an absolute value of $1 + \frac{1}{|\lambda|}$.

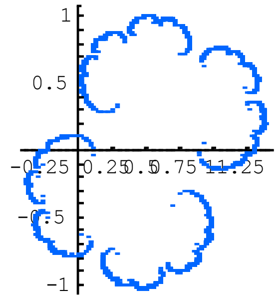


$\lambda =$ $2 + .2i$ $2 + .8i$ $2 + 1.3i$ $2 + 1.8i$

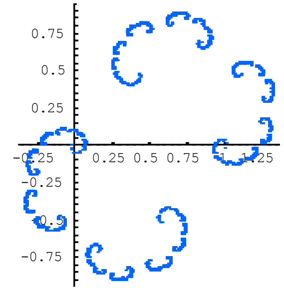
For **$\lambda = 1 + bi$**; where $b \in \mathbb{R}$.



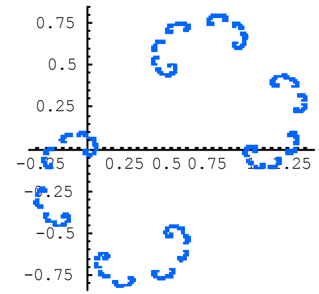
$\lambda =$ **1 + .2i**



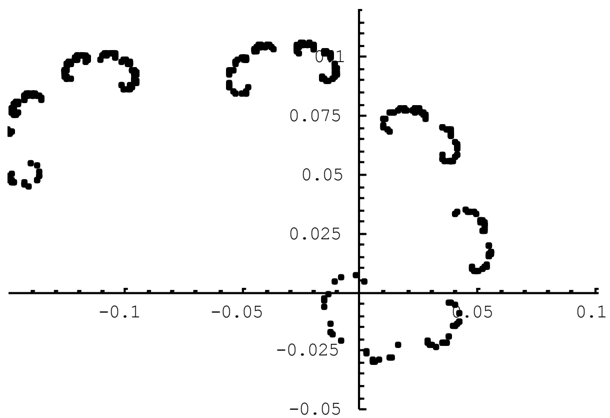
1 + .5i



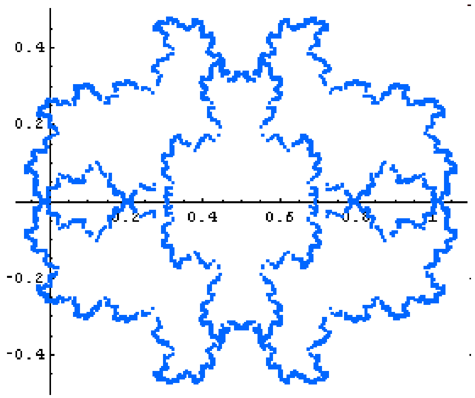
1 + .8i

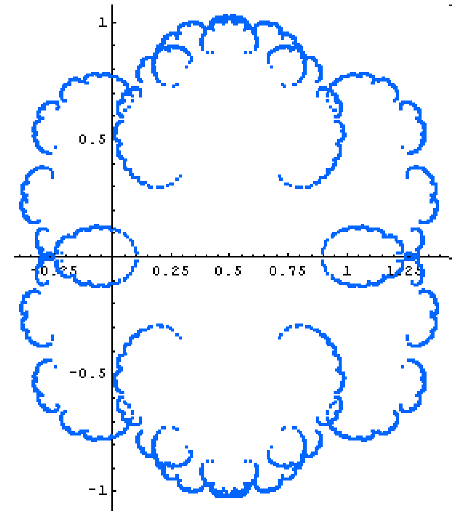


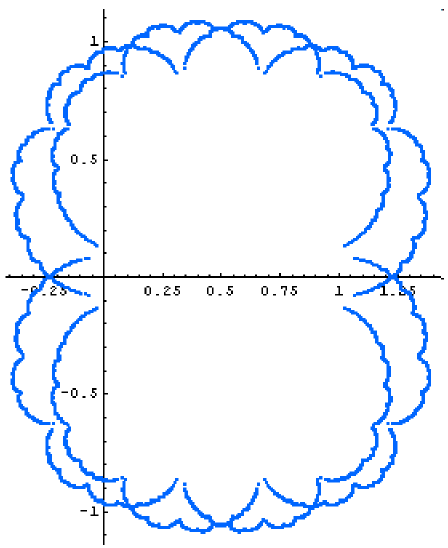
1+ i

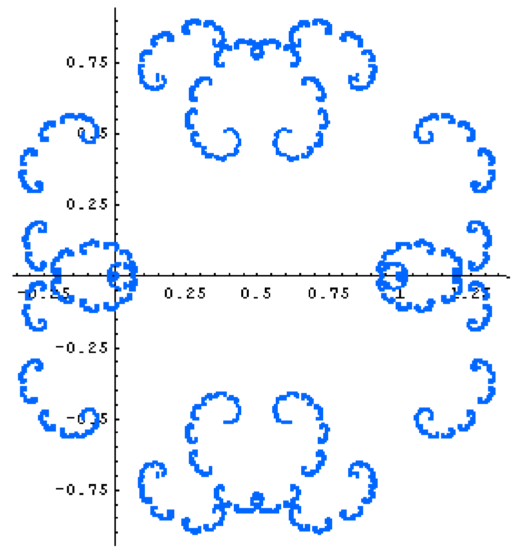


The orbits of some starting points z_0 in the z-plane tend toward the fixed point at infinity or toward the interior fixed point. When the inverse transformation of the logistic function is iterated, the interior points of the set converge on the Julia set boundary, and similarly the exterior points converge on the boundary.

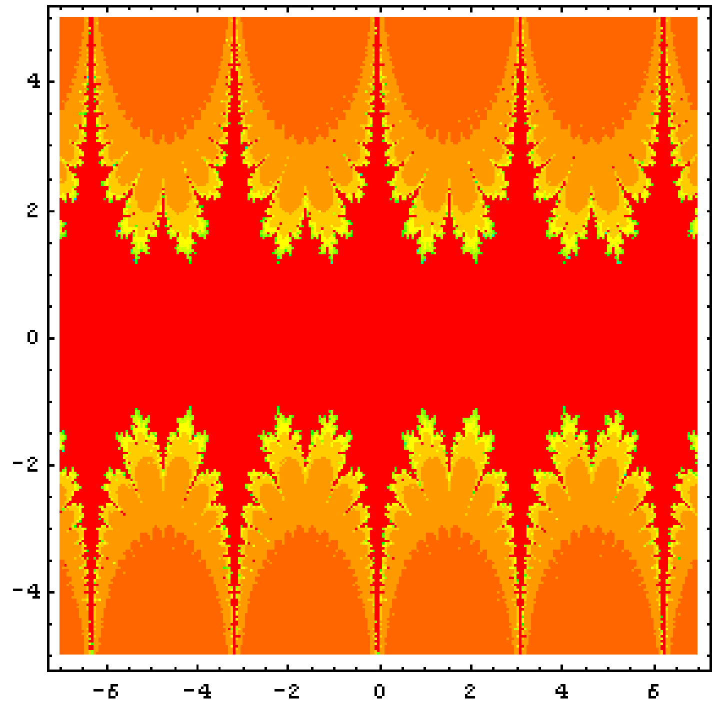

0.4
0.2
-0.2
-0.4
0.2
0.4
0.6
0.8
1


1
0.5
-0.5
-1
0.25
0.5
0.75
1
1.25

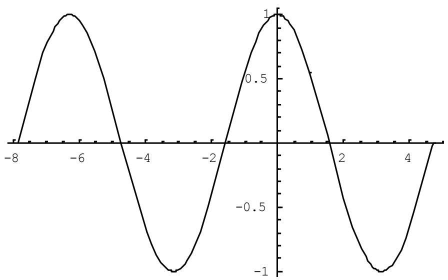

1
0.5
-0.5
-1
0.25
0.5
0.75
1
1.25


0.75
0.25
-0.25
-0.75
0.25
0.5
0.75
1
1.25

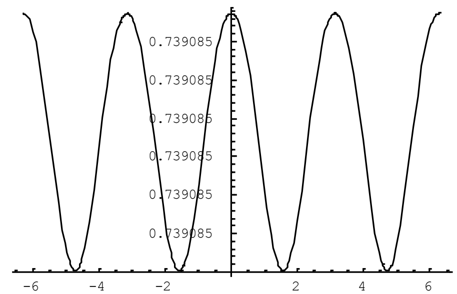
2- The fractal curve of Cosine function in complex number.



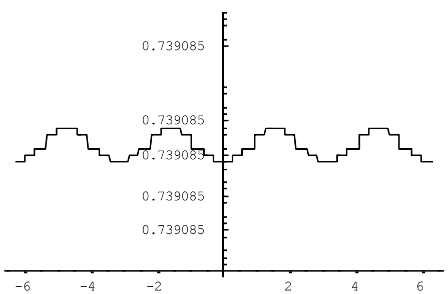
The behavior of the cosine function with a real value to iteration as the iteration increases (cos[cos[cos…[x]..]).



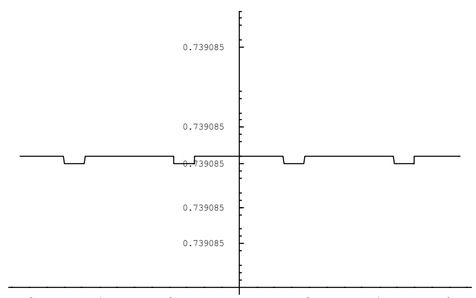
(a) $5\pi/2 < x < 3\pi/2$



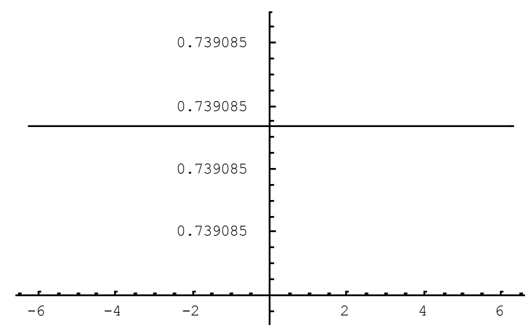
(b) 75 iteration



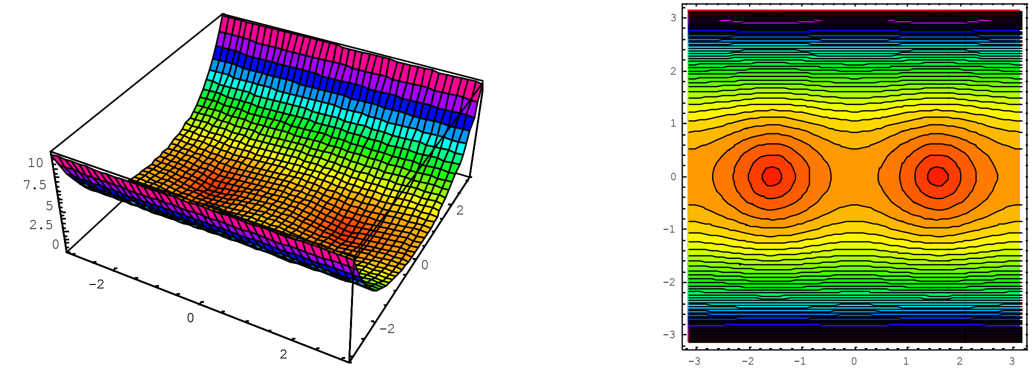
(c) 90



(d) 93

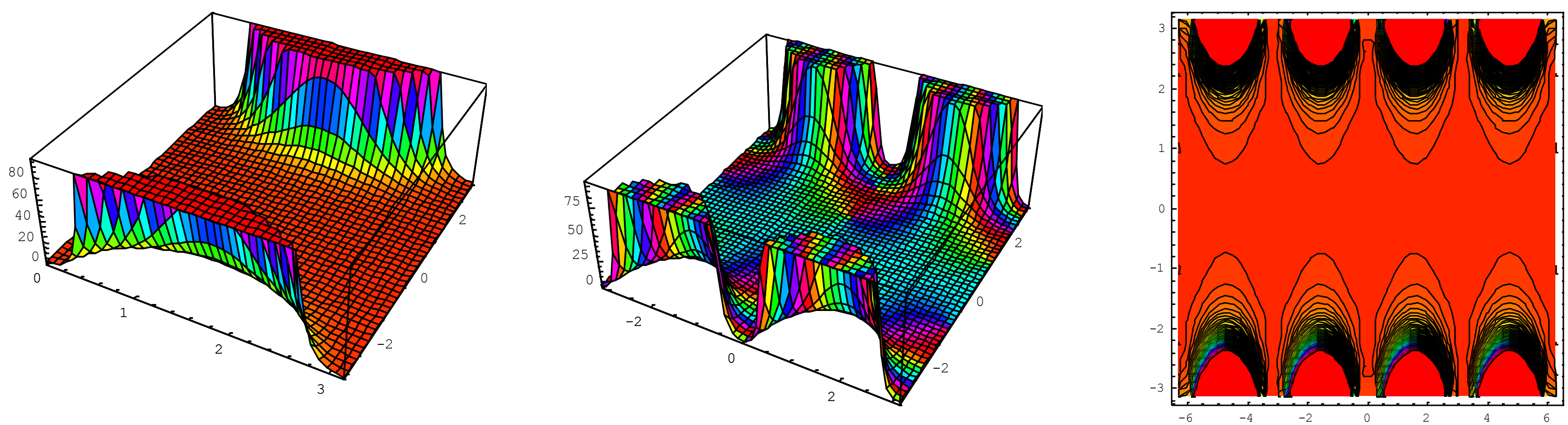


(e) 94 (.7390845 - .7390855)

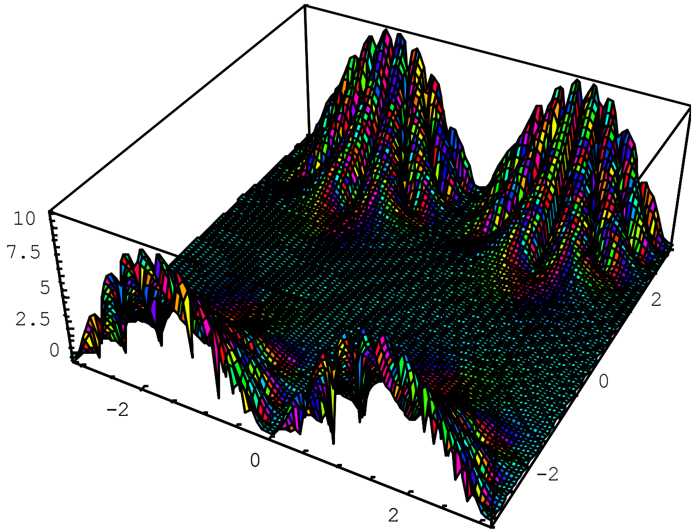


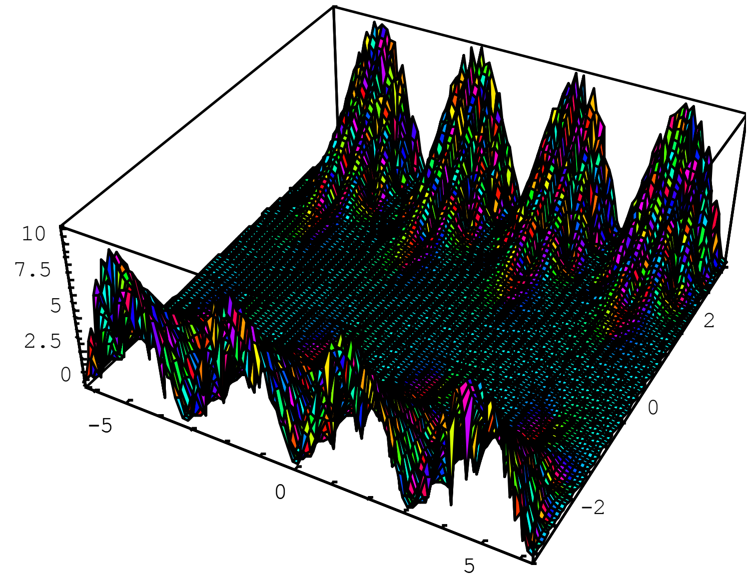
Plot for f(x) = cos(x) in 3-dimension and its contour.

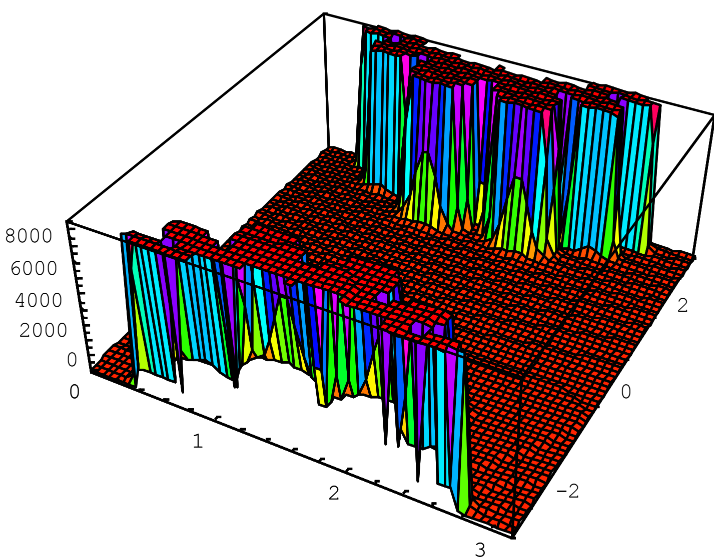
For the 2-iterations of the cosine function: cos(cos(x))

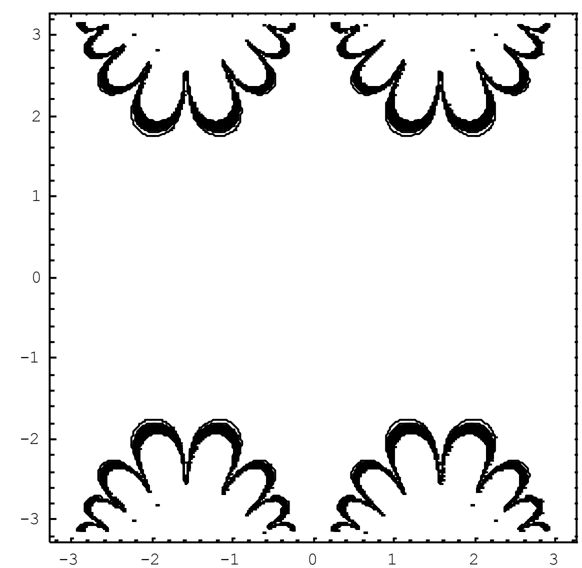


The behavior of the 3-iteration of cosine: cos(cos(cos(x))).

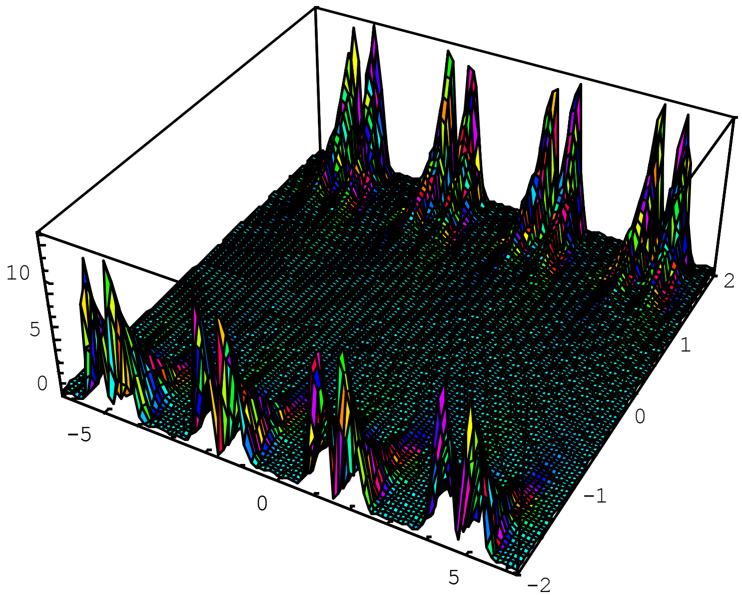


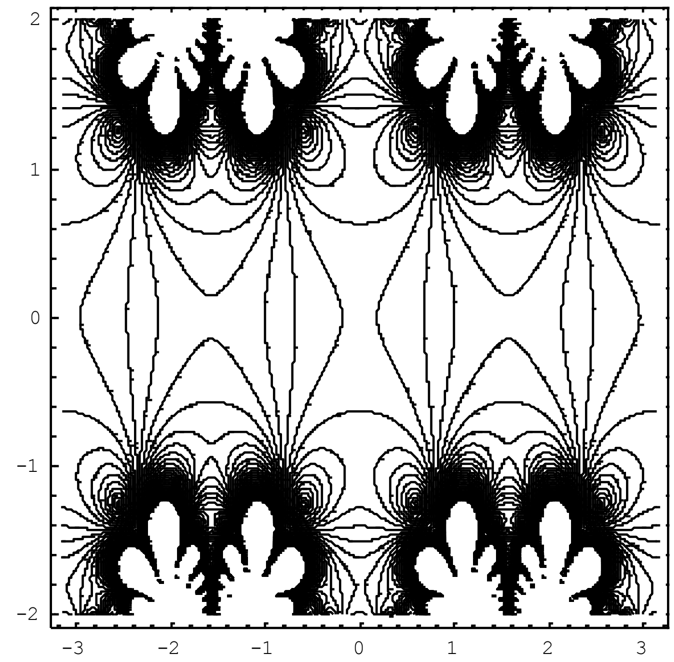




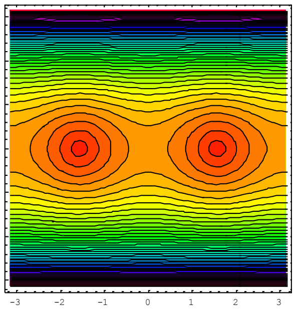


For 4-iterations for the cosine function: cos[cos[cos[cos[x]]]]

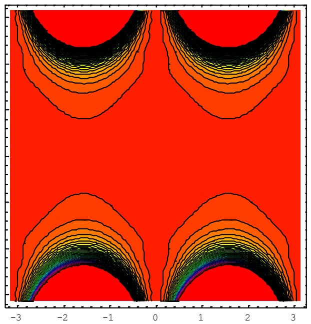




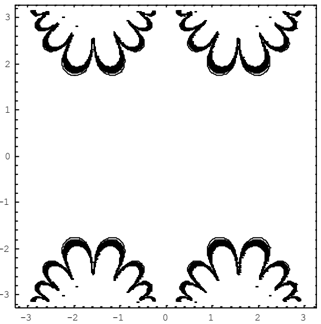
This splitting creates the fractal bifurcation to become more visible to us. If we increase the iteration more (nest more cos function), the dynamical behavior splits each in half and so on, in type of (divide into 2^{n-3}). The power (n-3) is when the fractals start occurring, the contour shows the chaos is starting to build-up.



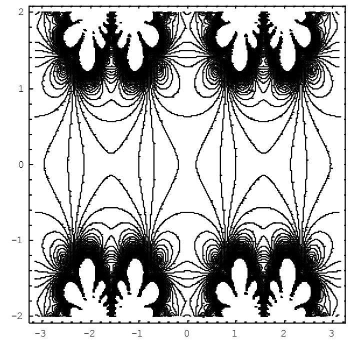
Cos[z]



Cos[Cos[z]]

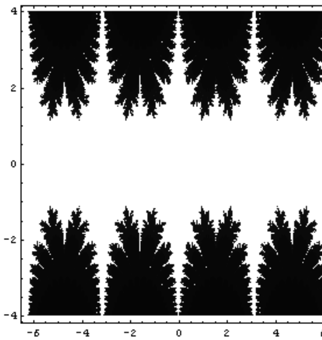


Cos[Cos[Cos[z]]]

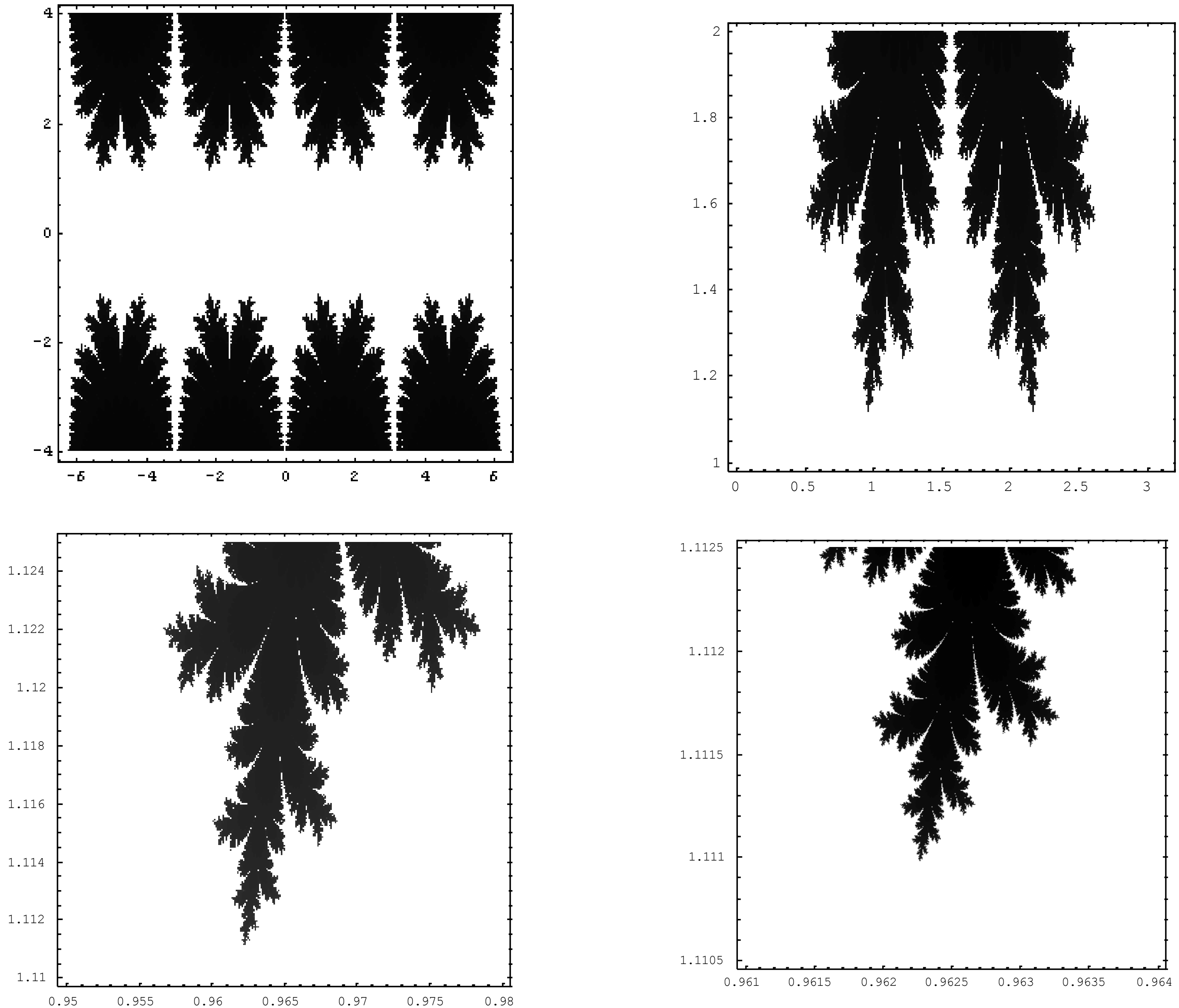


Cos[Cos[Cos[Cos[z]]]]

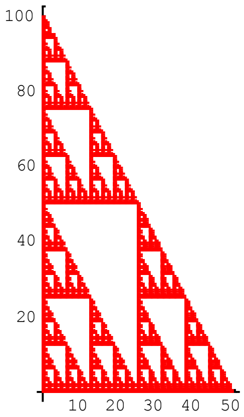
...



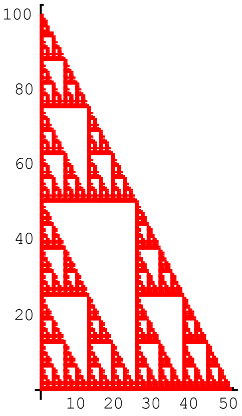
Cos[…[Cos[z]..]



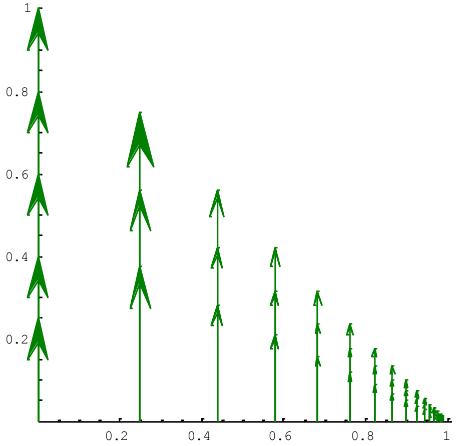
4
2
0
-2
-4
-6 -4 -2 0 2 4 6
2
1.8
1.6
1.4
1.2
1
0 0.5 1 1.5 2 2.5 3
1.124
1.122
1.12
1.118
1.116
1.114
1.112
1.11
0.95 0.955 0.96 0.965 0.97 0.975 0.98
1.1125
1.112
1.1115
1.111
1.1105
0.961 0.9615 0.962 0.9625 0.963 0.9635 0.964



3-



Generating Trees, a Sierpinski Triangle, and a Fern branch through Iterated Function System (I.F.S)



An Iterated Function System (I.F.S):

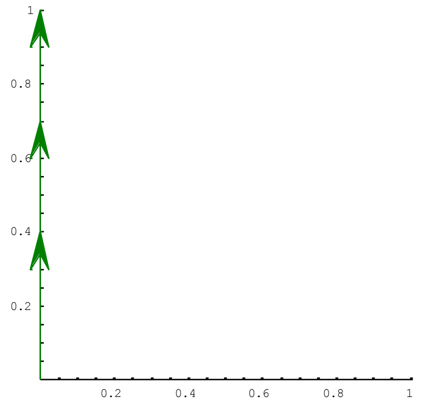
I.F.S specifies a discrete scattering dynamical system. An I.F.S consists of a set of contractive functions that define a more complex contractive function.

For Example:

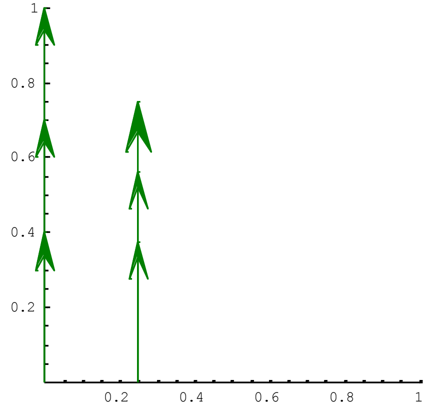
$$w\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}.75 & 0\\0 & .75\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0.25\\0\end{pmatrix} = \frac{3}{4}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1\\0\end{pmatrix}$$

Let T_0 be the original tree, then $T_n = W(T_{n-1})$ is the tree after n^{th} iterations.

Therefore; $\left(\bigcup_{n=0}^{N} T_n\right)$ is a set of trees lined up in the following with N = 15.

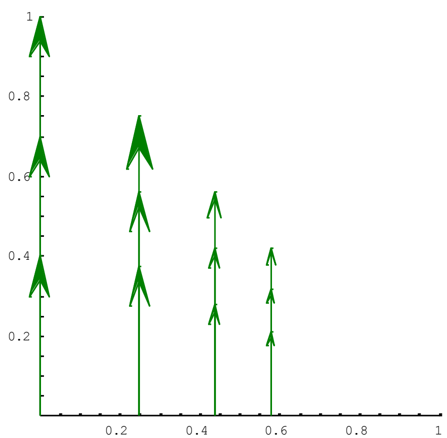


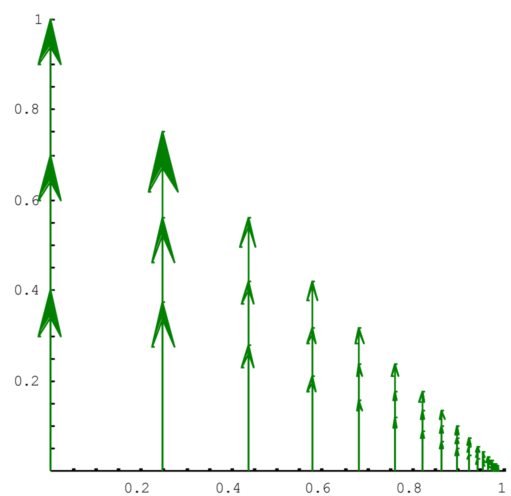
$$\begin{cases} .75x + .25 \\ .75\,y \end{cases}$$

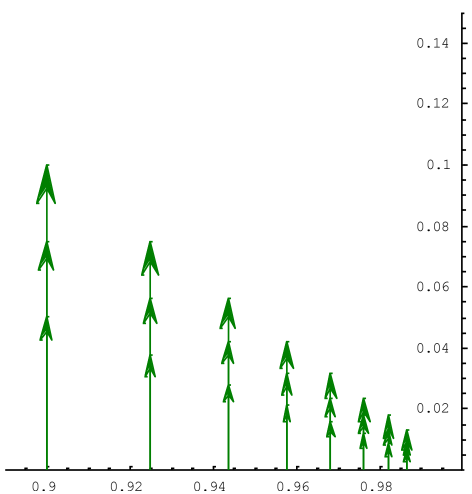


The union of the tree can be determine in the following:

$$\bigcup_{n=1}^{N} T_n = \sum_{n=1}^{N} \begin{pmatrix} 1-t_n & 0 \\ 0 & 1-t_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_n \\ 0 \end{pmatrix}$$

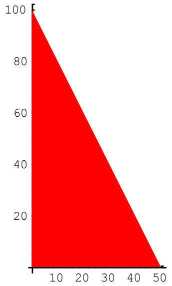






Part - B

Sierpinski triangle



In this part, the purpose is to create a Sierpinski triangle from a given triangle.

First we need to set the dimension and the coordinate values of the triangle and they are given by:

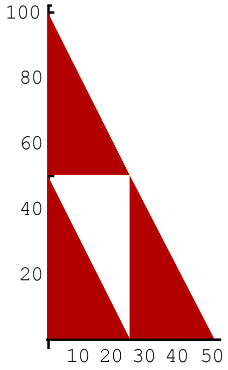
$$T_1\ (0,0) = (0,0)$$
$$T_2\ (50,0) = (50,0)$$
$$T_1\ (0,100) = (0,100)$$

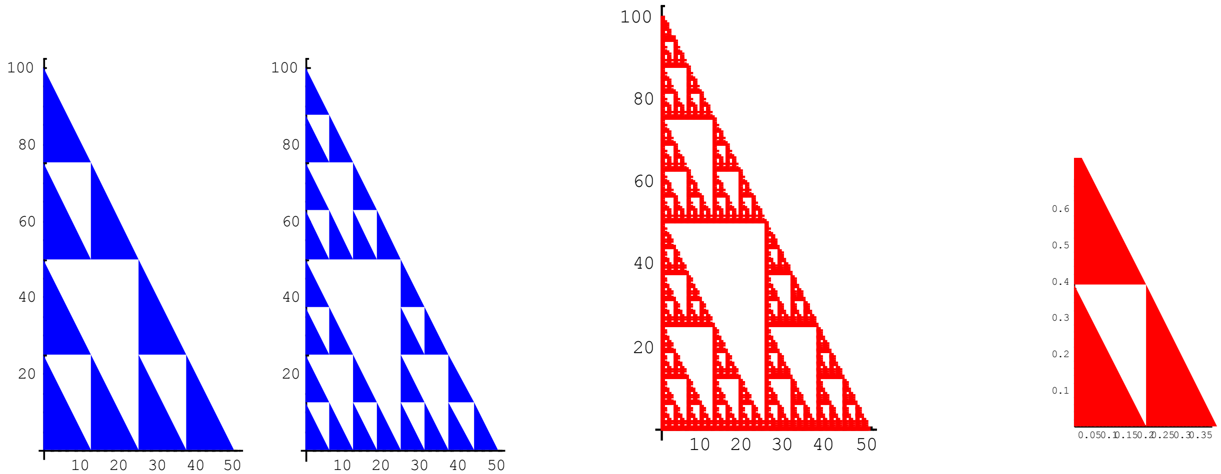
So we construct 3 smaller copies of the axiom triangle with dimension of ½ of the entire triangle.

$$T_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$T_5 = \begin{pmatrix} .5 & 0 \\ 0 & .5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 25 \\ 0 \end{pmatrix}$$

$$T_6 = \begin{pmatrix} .5 & 0 \\ 0 & .5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 50 \end{pmatrix}$$





Sierpinski gasket

There is a relation between the reduction factor „r“ and the number of pieces „t“ into which the structure can be divided.

That it will end up $t = \frac{1}{rD}$

The n-iteration $t = 3^n$ of side length $r = r_0 \left(\frac{1}{2}\right)^n$, where r_0 is the original length.

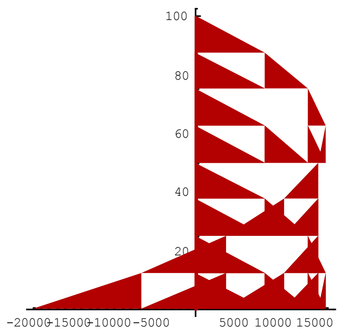
Therefore the fractal dimension D of Sierpinski gasket, by using log, is:

$$D = \frac{\log t}{\log \frac{1}{r}} = \frac{\log 3^n}{\log 2^n} = \frac{\log(\#\text{of triangles})}{\log(\text{magnification})} = \frac{\log 3}{\log 2} = 1.58496$$

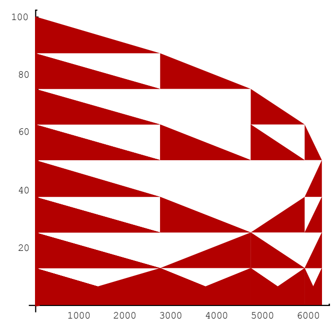
Then transpose the linear transformation of the Sierpinski triangle through a polynomial mapping P.Where the mapping is given by the function:

$$P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax(x - b) \\ y \end{pmatrix}$$

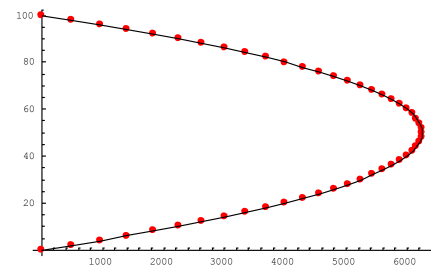
Mapping to a second-degree function: ax(x - b), and y is mapping to itself $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} ax_{n-1}(x_{n-1} - b) \\ y_{n-1} \end{pmatrix}$



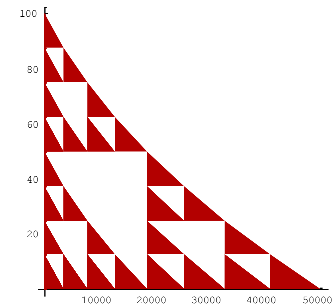
a = -500 , b = 50



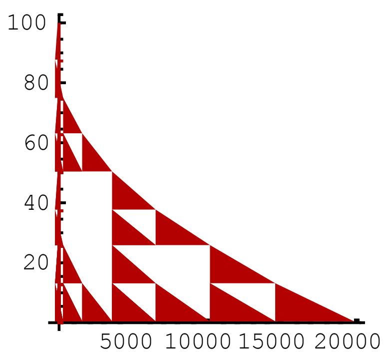
a = -10 , b = 50

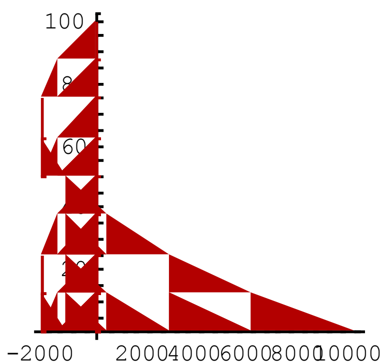


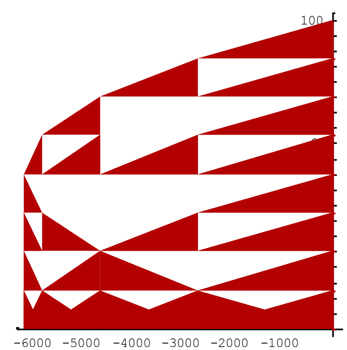
a = -10 , b = 50

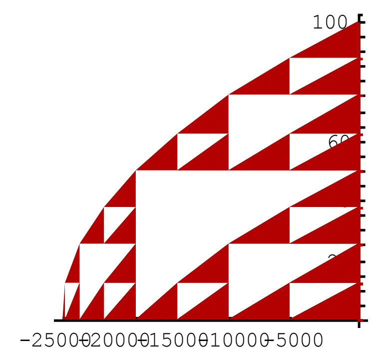


a = 10 ; b = -50







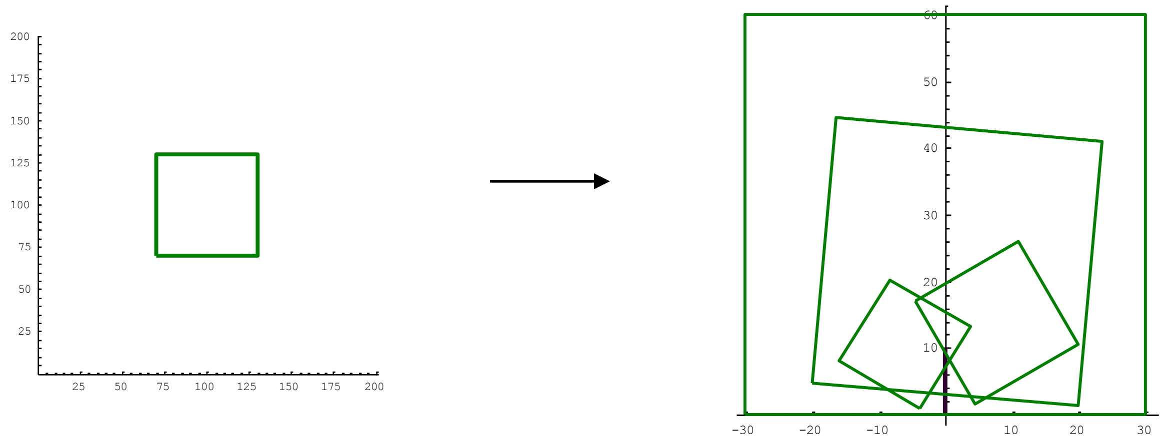


a = 10 , b = 20, 40, 50, 80

Part C -

FERN

A sequence of sets A_n in $\wp^2$ that converges to a fern-like set through a proper I.F.S.



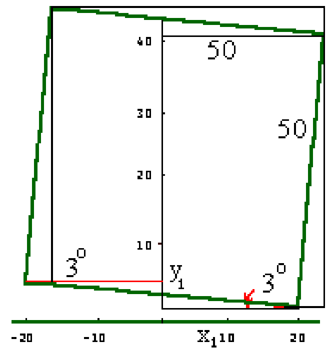
The matrices give the general equation for the system: $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$

Can be reform or rewritten in a polar coordinate format, which is describing the affined transformation

for I.F.S as follows: $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r.\cos\theta & s.\sin\phi \\ -r.\sin\theta & s.\cos\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$

The I.F.S for the square in fig.12-a is given by: $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 60 & 0 \\ 0 & 60 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 70 \\ 70 \end{pmatrix}$

$2x_1 = 50\cos(3) = 49.9 \Rightarrow x_1 = 24.97$,
and $y = 50\sin(3) = 2.62 \Rightarrow y_1 = 1.67 + ½\,(2.62) = 2.98$
$\Rightarrow (x_0, y_0) = (-24.97, 2.98)$,



$x_1 = 24.97$, $y_2 = 1.67 - 24.97\sin(3) = .3 \Rightarrow (24.97, .37)$
$x_2 = x_1 + 50\sin(3) = 27.6$, $y_2 = y_1 + 50\cos(3) = 50.23 \Rightarrow (27.6, 50.23)$
$x_3 = x_2 - 50\cos(3) = -22.33$, $y_3 = y_2 + 50\sin(3) = 52.85 \Rightarrow (-22.3, 52.85)$

F_1 = (0,0), (0,10)

F_2 = (-24.97, 2.98), (24.97, .37), (27.6, 50.2), (-22.3, 52.85)

F_3 = (-6, -5.3), (6, 8.5), (-9.6, 21.7), (-21.6, 7.97)

F_4 = (4.5, -7.36), (-4.5, 8.24), (12.3, 22.64), (21.3, 7.04)

Since the I.F.S used affine transformations, which produce the fern from contractions. The four equations F_1, F_2, F_3, and F_4 can be rewritten in other format (form contractions). Each transformation is of the form of matrices or vectors:

$$F_{new} = A\,F_{old} + B$$

where A (2x2) and B ((e, f) constant vector) are the matrices, F_{new} is the new values for the system in I.F.S, and F_{old} the domain of the mapping. Therefore, the contraction may be written in form:

$$\begin{cases} x_1 = a\,x_0 + b\,y_0 + e \\ y_1 = c\,x_0 + d\,y_0 + f \end{cases} \qquad \text{for } -30 \le x_0 \le 30, \text{ and } 0 \le y_0 \le 60.$$

The contraction for F_1 is:

$$\Rightarrow \begin{cases} -30\,a + e = 0 \\ -30\,c + f = 0 \end{cases} \text{ and } \begin{cases} 30\,a + e = 0 \\ 30\,c + f = 0 \end{cases} \Rightarrow a = c = e = f = 0$$

$$\Rightarrow \begin{cases} 60\,b = 0 \\ 60\,d = 10 \end{cases} \Rightarrow b = 0\,,\, d = \frac{10}{60} = .1667$$

$$\Rightarrow \begin{cases} x_1 = 0\,x_0 + 0\,y_0 \\ y_1 = 0\,x_0 + .1667\,y_0 \end{cases}$$

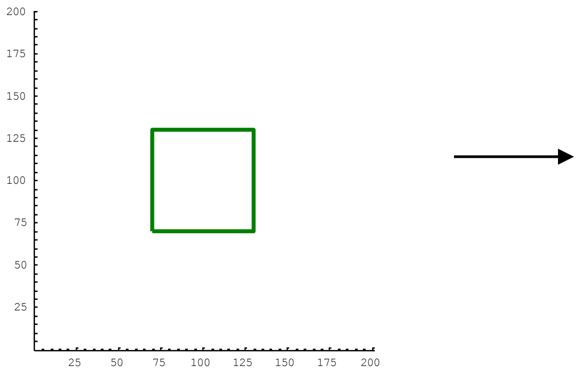
The contraction for F_2 : $\begin{cases} x_1 = .83\,x_0 + .045\,y_0 \\ y_1 = -.043\,x_0 + .83\,y_0 + 1.67 \end{cases}$

F_3: $\begin{cases} x_1 = .2\,x_0 - .26\,y_0 \\ y_1 = .23\,x_0 + .22\,y_0 + 1.67 \end{cases}$

F_4: $\begin{cases} x_1 = -.15\,x_0 + 0.28\,y_0 \\ y_1 = .26\,x_0 + .24\,y_0 + .44 \end{cases}$

The dimension D of the attractor A_∞ can be computed from the equation $N.a^D = 1$. Where N is the number of the mapping of the square (N = 4), and a is the percentage of the 4 reduced replica by (a = ¾). Therefore; the self-similarity dimension D is:

$$D = \frac{\log N}{\log \frac{1}{a}} = \frac{\log 4}{\log \frac{1}{3/4}} = \frac{\log 4}{\log \frac{4}{3}} = 4.818$$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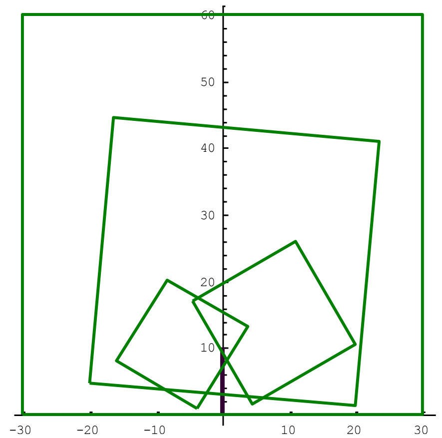
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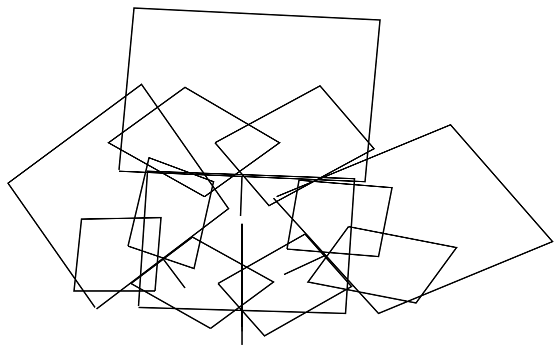

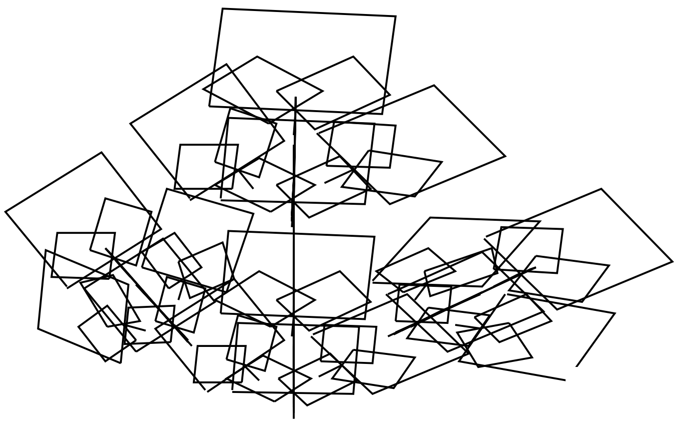

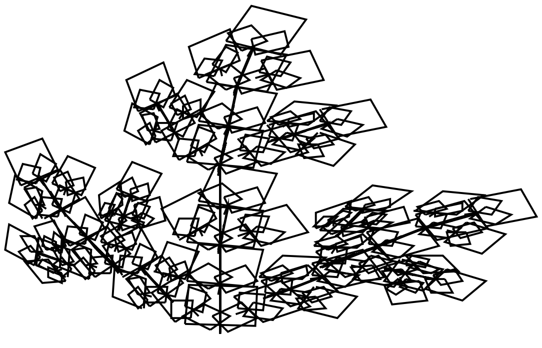

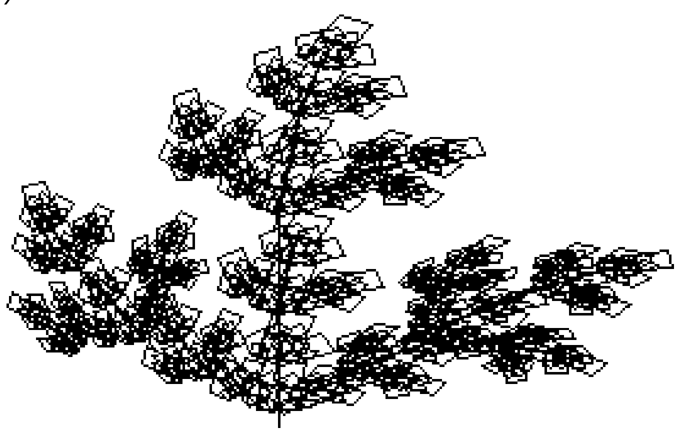

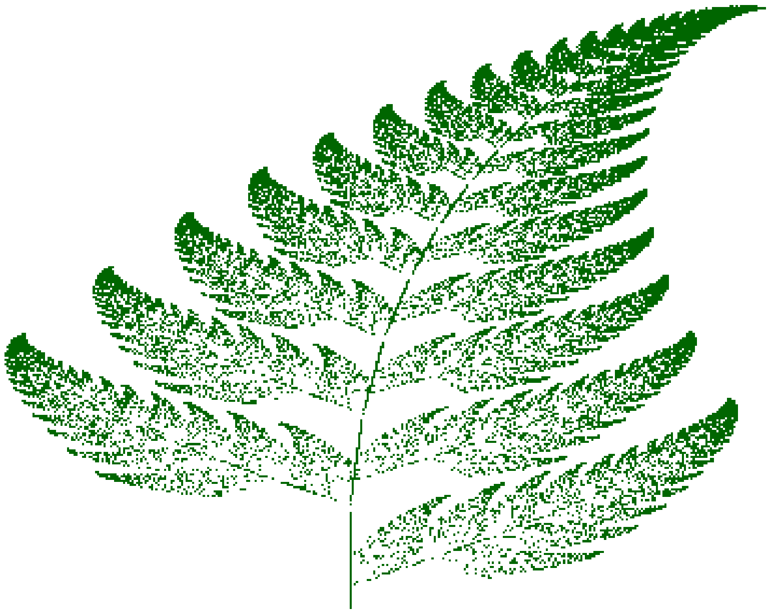

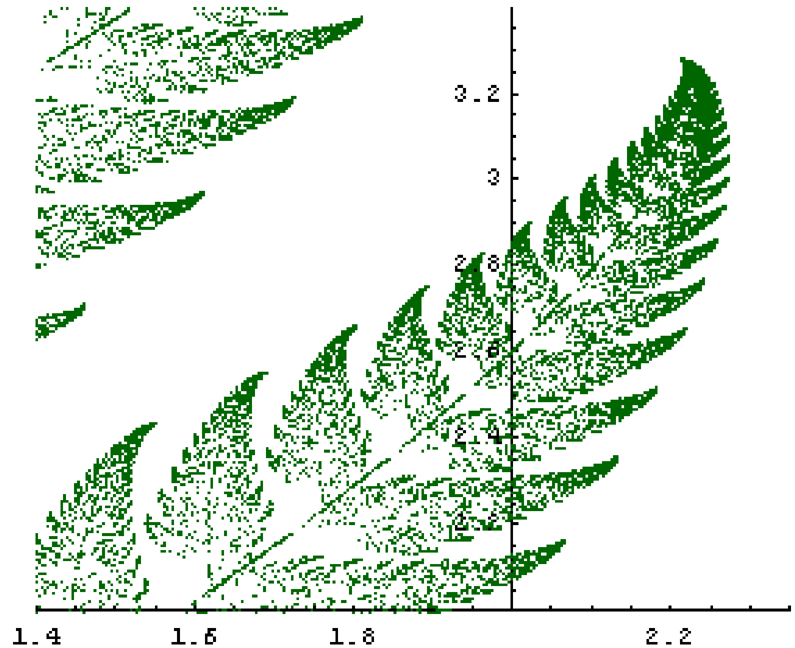

Barnsley's fern after n-iterations

"This corner looks like the leaf, if only I squeeze it and distort it and turn it about. This piece is a distortion of the whole thing."

Barnsley, Michael.

Special Thanks

To: Dr. Shiau, Lie June.
Jesse Warren.